# Höhere Algebra (Säule II)

Übung, LVA 405.451
C. Fuchs, R. Paulin

## 9. Übungsblatt, WS 2013/14 — 20.12.2013

It suffices to solve 6 out of 8. Feel free to solve more in order to collect some extra points! Below, $p$ always denotes a prime number.

1. Let $K$ be a field of characteristic $p$ and let $n$ be a positive integer. Prove that an element $\alpha \in K$ has at most one $p^n$-th root in any extension of $K$.

2. Calculate the degree and separable degree of the field extension $K(\alpha) \supseteq K$, where $K = \mathbb{F}_p(X)$, and $\alpha$ is a zero of the irreducible polynomial $f(T) = T^{p^3} - XT^p - 1 \in K[T]$. (Hint: To check that $f$ is irreducible, note that $f \in \mathbb{F}_p[X][T] = \mathbb{F}_p[T][X]$, and $f$ has degree 1 in $X$.)

3. Let $E \supseteq K$ be an algebraic extension. Prove that the separable closure of $K$ in $E$ is a field.

4. Show that if $E \supseteq K$ is a finite extension, say $E = K(\alpha_1, \ldots, \alpha_n)$, where each $\alpha_i$ is separable over $K$, then $E \supseteq K$ is separable.

5. Prove that any algebraic extension of $\mathbb{Q}$ is separable.

6. Let $L \supseteq K$ be a finite field extension, where $\mathrm{char}(K) = p$ and $\gcd([L : K], p) = 1$. Show that $L \supseteq K$ is a separable extension.

7. Find a primitive element for the field extension $\mathbb{Q}(\sqrt{5}, \sqrt[3]{7}) \supseteq \mathbb{Q}$.

8. Give a self-contained proof of the fact that the multiplicative group of a finite field is cyclic.