Höhere Algebra (Säule II)

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1. Let α, β be algebraic over some field K with minimal polynomials f, g respectively. Prove that $\mathrm{Res}_Y(f(X-Y), g(Y))$ and $\mathrm{Res}_Y(Y^n f(X/Y), g(Y))$, $n = \deg f$, are polynomials in X with coefficients in K and which have $\alpha + \beta$ respectively $\alpha\beta$ as a root. Use this to calculate the minimal polynomial of $\sqrt{2} + \sqrt{5}$.

2. Show that if $L \supseteq K$ is an algebraic extension, and K is algebraically closed, then $L = K$.

3. Prove that $\mathbb{Q}(e^3)$ is isomorphic to $\mathbb{Q}(1 + \pi^2)$, where $e = \exp(1)$. Prove that $\mathbb{Q}(\sqrt{2})$ is not isomorphic to $\mathbb{Q}(\sqrt{3})$.

4. Calculate the minimal polynomial over $\mathbb{Q}$ of $\zeta_p = \exp(2\pi i/p) \in \mathbb{C}$ for p a prime. What is $\deg(\zeta_p)$? What are the $\mathbb{Q}$-conjugates of ζ_p. The field $\mathbb{Q}(\zeta_p) =: \mathbb{Q}[p]$ is called the cyclotomic field over $\mathbb{Q}$ of level p.

5. Calculate the $\mathbb{Q}$-conjugates in $\mathbb{C}$ of $\sqrt{2} + \sqrt{5} + \sqrt[3]{2}$.

6. a) Describe the set $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2}), \mathbb{C})$ and the automorphism group $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2})) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[4]{2}), \mathbb{Q}(\sqrt[4]{2}))$. b) Prove that the fields $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(\sqrt[4]{2}i)$ are $\mathbb{Q}$-isomorphic. Are they equal?