Höhere Algebra (Säule II)

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1. Find the minimal polynomial of $\alpha = \sqrt{5} + \sqrt{7}$ over $\mathbb{Q}$. Give a basis of $\mathbb{Q}(\alpha) \supseteq \mathbb{Q}$.
2. Show that $\mathbb{Q}(\sqrt{2} + \sqrt[3]{3}) = \mathbb{Q}(\sqrt{2}, \sqrt[3]{3})$.
3. Let $K(\alpha) \supseteq K$ be a finite field extension. Show that if $[K(\alpha) : K]$ is odd, then $K(\alpha^2) = K(\alpha)$. (Easy extra exercise: find a counterexample for this if $[K(\alpha) : K]$ is even.)
4. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 3. Show that the degree of the splitting field of f over $\mathbb{Q}$ is $1, 2, 3$ or 6. Find an example for each of these cases. (The splitting field of f is the extension of $\mathbb{Q}$ generated by the complex zeros of f.)
5. Check that $X^3 + X + 1$ is an irreducible polynomial in $\mathbb{F}_2[X]$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. Verify that $\mathbb{F}_8 = \mathbb{F}_2[X]/(X^3 + X + 1)$ is a field. Check that $\mathbb{F}_8^\times$ is a cyclic group of order 7.
6. Let $L \supseteq K$ be an algebraic extension, and let $K \subseteq R \subseteq L$ be an intermediate ring. Show that R is a field.
7. Let $L \supseteq K$ be a field extension, and let E and F be subfields of L such that $K \subseteq E, F$. Show that if $[E : K]$ and $[F : K]$ are finite and coprime, then $[EF : K] = [E : K] \cdot [F : K]$.
8. Let $\mathbb{Q}(x) \supseteq \mathbb{Q}$ be a field extension with x transcendental over $\mathbb{Q}$, and let $y = f(x)/g(x) \in \mathbb{Q}(x)$, where $f, g \in \mathbb{Q}[x]$, $g \neq 0$ and $(f, g) = 1$. Show that $[\mathbb{Q}(x) : \mathbb{Q}(y)] = \max(\deg f, \deg g)$.