Höhere Algebra (Säule II)

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1. Let R be a domain, and $f, g \in R[X]$ such that $\deg g > 0$. Then $f(g(X))$ is irreducible $\Rightarrow f(X)$ is irreducible. (Special case: $f(X+c)$ is irreducible $\Leftrightarrow f(X)$ is irreducible.)

2. Let $F(X,Y) \in R[X,Y]$. Show that if $F(X,X) = 0 \in R[X]$, then $X - Y \mid F(X,Y)$. More generally, show that if $F(X,Y) \in R[X,Y]$ and $g(X) \in R[X]$ such that $F(X, g(X)) = 0 \in R[X]$, then $Y - g(X) \mid F(X,Y)$.

3. Calculate the gcd of $X^4 + 2X$ and $X^2 + 5$ in $\mathbb{C}[X]$ and in $\mathbb{F}_3[X]$. (Here $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.)

4. We define the reciprocal polynomial of $f(X) = a_n X^n + \cdots + a_0 \in R[X]$ (where R is a domain and $a_0, a_n \neq 0$) as $f^*(X) = a_0 X^n + \cdots + a_n$. Show that $f(X)$ is irreducible $\Leftrightarrow f^*(X)$ is irreducible.

5. Prove that the polynomials $X^7 + 48X - 24$ and $X^4 + 4X^3 + 6X^2 + 4X + 3$ are irreducible in $\mathbb{Q}[X]$. (Hint: the Eisenstein criterion and exercise 1 might be useful.)

6. Let $f \in \mathbb{Z}[X]$ be a monic polynomial, p a prime number, and $\overline{f} \in (\mathbb{Z}/p\mathbb{Z})[X]$ the reduction of f mod p. Show that if $\overline{f}$ is irreducible, then f is irreducible.

7. Let $f, g \in \mathbb{Q}[X]$, where g is irreducible. Show that $g^2 \mid f \Leftrightarrow g \mid f, f'$.

8. Let $f(X) = X^3 - 2$ and $g(X) = X^2 - X + 4$. Note that $f(X)$ is irreducible in $\mathbb{Q}[X]$, and $\mathbb{Q}[X]$ is a PID, so the ideal $(f(X)) \subseteq \mathbb{Q}[X]$ is a maximal ideal, hence $K = \mathbb{Q}[X]/(f(X))$ is a field. Let $\phi\colon \mathbb{Q}[X] \to \mathbb{Q}[X]/(f(X))$ be the quotient map, and let $x = \phi(X)$. Calculate the inverse of $\phi(g)$ in K (so find a polynomial $h \in \mathbb{Q}[X]$ such that $\phi(h) = \phi(g)^{-1}$).