Höhere Algebra (Säule II)

Übung, LVA 405.451

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1. Recall that a ring homomorphism $\varphi\colon R \to S$ is of finite type (or in other words, S is a finitely generated R-algebra), if there is a surjective homomorphism $R[X_1, \ldots, X_n] \to S$ extending φ. Prove that if $f\colon A \to B$ and $g\colon B \to C$ are homomorphisms of finite type, then $g \circ f\colon A \to C$ is also of finite type. (An easy extra exercise: $g \circ f$ is of finite type $\Rightarrow$ g is of finite type.)

2. Let $\varphi\colon R[X_1, \ldots, X_n] \to S$ be a surjective ring homomorphism with kernel I. Suppose $I = I_0 R[X_1, \ldots, X_n]$ for some ideal $I_0 \subseteq R$. Prove that $I_0 = \ker(\varphi|_R)$ and $S \cong \varphi(R)[X_1, \ldots, X_n]$.

3. Express the symmetric polynomials $\sum_{i \neq j} X_i^2 X_j$ and $\sum_{i \neq j} X_i^3 X_j$ (where $i, j \in \{1, \ldots, n\}$) as polynomials of the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$.

4. Calculate the discriminant of $f(X) = (X+1)(X-2)(X-3) \in \mathbb{Z}[X]$ using the two different definitions.

5. Calculate the discriminant of $X^n - a \in R[X]$, where $n \geq 1$ and $a \in R$.

6. Show that $\operatorname{Res}(fg, h) = \operatorname{Res}(f, h) \cdot \operatorname{Res}(g, h)$ for every nonzero $f, g, h \in R[X]$ (where R is a domain).

7. Let $f(X) = \prod_{i=1}^{4}(X - \alpha_i)$ and $g(X) = \prod_{j=1}^{3}(X - \beta_j)$, where $\beta_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4$, $\beta_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4$, $\beta_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3$. Note that g is invariant under the permutations of $\alpha_1, \ldots, \alpha_4$. Express the coefficients of g using the coefficients of f. (The polynomial g is called the resolvent of the quartic polynomial f.)

8. Let $R[X_1, \ldots, X_n]$ be a polynomial ring, and let $\sigma_1, \ldots, \sigma_n$ be the elementary symmetric polynomials of $X_1, \ldots, X_n$. Let $p_k = X_1^k + \cdots + X_n^k$ for every $k \in \mathbb{Z}_{\geq 1}$. Show the following identities (these are called Newton's identities):

$$p_k - \sigma_1 p_{k-1} + \sigma_2 p_{k-2} - \cdots + (-1)^{k-1}\sigma_{k-1} p_1 + (-1)^k k \sigma_k = 0, \quad \text{if } 1 \leq k \leq n,$$
$$p_k - \sigma_1 p_{k-1} + \sigma_2 p_{k-2} - \cdots + (-1)^n \sigma_n p_{k-n} = 0, \quad \text{if } k > n.$$

(Warning: In the last term of the first equation's left hand side, we have $(-1)^k k\sigma_k$, and not $(-1)^k n\sigma_k$! One might expect the term $(-1)^k n\sigma_k$, because it is natural to define $p_0 = X_1^0 + \cdots + X_n^0 = n$, and then $(-1)^k \sigma_k p_0 = (-1)^k n\sigma_k$.)