Höhere Algebra (Säule II)

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1. Let R be a domain and $a, b, c \in R, c \neq 0$. Assume that $\gcd(ac, bc)$ exists. Prove that then $\gcd(a, b)$ also exists and that $c \cdot \gcd(a, b)$ is associated to $\gcd(ac, bc)$.

2. Let $R = \mathbb{Z}[\sqrt{-7}] := \{a + b\sqrt{-7}; a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$. Verify that R together with the multiplication and addition of $\mathbb{C}$ forms a domain which is not a UFD.

3. Calculate the gcd of $a = 135 - 14i$ and $b = 155 + 34i$ in $\mathbb{Z}[i]$ and express the gcd in the form $ax + by$ with $x, y \in \mathbb{Z}[i]$.

4. Show that $\mathbb{Z}[\sqrt{2}]$ is a euclidean ring with euclidean norm function $N(a + b\sqrt{2}) = |a^2 - 2b^2|$.

5. Let $f = 2X^2 - 1, g = X^4 + 2X^2 + 1 \in \mathbb{Z}[X]$. Write f, g as elements in $\Gamma(\mathbb{N}, \mathbb{Z})$ by giving the support and the values on the support. Calculate $-Xfg$ and $3f - g$ by using the definition of the operations and formulate the answer also in the usual notation. Do the same for $f = 2X^2 - 1, g = Y^4 + 2Y^2 + 1 \in \mathbb{Z}[X, Y]$.

6. Let R be a domain. Show that $R[X]^\times = R^\times$.

7. Let R be a unitary (not necessarily commutative) ring. The center of R is $Z(R) = \{x \in R; xy = yx \text{ for every } y \in R\}$. Prove that $Z(R)$ is a subring of R, and that $Z(R[X]) = Z(R)[X]$.

8. Prove that there is a ring isomorphism $R[X]/(aX - 1) \cong S_a^{-1}R$ for every $a \in R$, where $S_a = \{1, a, a^2, a^3, \dots\}$. (Hint: Define inverse homomorphisms using the universal property of $R[X]$ and $S_a^{-1}R$.)