Höhere Algebra (Säule II)

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In the following exercises R is a commutative unitary ring, and $S \subseteq R$ is a multiplicative subset.

1. Show that if $Q \subseteq R$ is a primary ideal, then $\sqrt{Q}$ is a prime ideal.

2. Show that $\iota_S\colon R \to S^{-1}R$, $r \mapsto r/1$ is a ring homomorphism. Show further that $S^{-1}R = 0 \Leftrightarrow 0 \in S$ and that ι_S is injective $\Leftrightarrow$ S does not contain a zero divisor.

3. Show that there is a bijection between the prime ideals $\mathfrak{P}$ of $S^{-1}R$ and the prime ideals $\mathfrak{p}$ of R contained in $R \setminus S$.

4. Let $\overline{S} = \{a \in R;\ \exists b \in R\colon ab \in S\}$. Show that $\overline{S}$ is a multiplicative subset of R, $S \subseteq \overline{S}$, and
$$S^{-1}R \cong \overline{S}^{-1}R.$$

5. Let $\varphi\colon R \to R'$ be a ring homomorphism. Check that $\varphi(S)$ is a multiplicative subset of R'. Prove that $\varphi_S\colon S^{-1}R \to \varphi(S)^{-1}R'$, $r/s \mapsto \varphi(r)/\varphi(s)$ is a ring homomorphism, and that if φ is injective/surjective, then φ_S is injective/surjective.

6. Show that
$$R \to \prod_{\mathfrak{p}} R_{\mathfrak{p}}, \quad r \mapsto \left(\ldots, \frac{r}{1}, \ldots\right)$$
is an injective ring homomorphism, where $\mathfrak{p}$ runs through the maximal ideals of R. Conclude that R is reduced if and only if $R_{\mathfrak{p}}$ is reduced for every maximal ideal $\mathfrak{p}$. (A ring R is reduced, if $\sqrt{(0)} = (0)$ in R.)

7. Let I be an ideal of R. Prove that $S^{-1}R/(S^{-1}I) \cong ((S+I)/I)^{-1}(R/I)$.

8. Let $p_1, \ldots, p_n$ be primes in $\mathbb{Z}$ and S the multiplicative set generates by them. Give a description of $S^{-1}\mathbb{Z}$.