Höhere Algebra (Säule II)

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1. Let $R = \{f\colon \mathbb{Z}_{>0} \to Q\}$ be the ring of functions mapping $\mathbb{Z}_{>0}$ into $\mathbb{Q}$. We say that $f \in R$ is a Cauchy-function, if for every $\varepsilon \in \mathbb{Q}_{>0}$ there is an N such that $|f(m) - f(n)| < \varepsilon$ for every $m, n > N$. We say that $f \in R$ is a null-function, if $\lim_{n\to\infty} f(n) = 0$. Denote the set of Cauchy-functions by C, and the set of null-functions by I. Prove that

   a) C is a subring of R;
   b) I is an ideal of C, but not of R;
   c) there is a ring isomorphism $C/I \cong \mathbb{R}$.

2. Let R be a ring and I an ideal of R. Prove that there is a bijection between the ideals J of R containing I and the ideals of R/I given by the map $J \mapsto J/I = \{j+I; j \in J\}$.

3. Let R be a ring, I, J be ideals of R with $I \subseteq J$. Show that there is an isomorphism $(R/I)/(J/I) \cong R/J$.

4. Let R be a ring, $R' \subseteq R$ be a subring, and I be an ideal of R. Prove that $R' \cap I$ is an ideal of R', and $R' + I = \{a + b;\, a \in R',\, b \in I\}$ is a subring of R. Also prove that $R'/(R' \cap I) \cong (R' + I)/I$.

In the next exercises let R be a commutative unitary ring.

5. a) Prove that if I, J are ideals of R, then $IJ \subseteq I \cap J$. Give an example where $IJ \neq I \cap J$. b) Prove that if $\mathfrak{p}$ is a prime ideal of R, and $I_1, \ldots, I_n \subseteq R$ are ideals, then $\prod_{k=1}^{n} I_k \subseteq \mathfrak{p}$ implies that $I_k \subseteq \mathfrak{p}$ for some $k \in \{1, \ldots, n\}$.

6. Show that: a) If $I + J = R$ for some ideals $I, J \subseteq R$, then $IJ = I \cap J$. b) If $I_1, \ldots, I_n$ are ideals of R with $I_i + I_j = R$ for all $i \neq j$, then $I_1 \cdots I_n = I_1 \cap \cdots \cap I_n$.

7. Let $I \subseteq R$ be an ideal and $S \subseteq R$ a subset. Then we define the *ideal quotient* $(I : S) = \{r \in R;\, rs \in I \text{ for every } s \in S\}$. Prove that

   a) $(I : S)$ is indeed an ideal of R, and $I \subseteq (I : S)$.
   b) $(I : S) = (I : (S))$, where (S) denotes the ideal generated by S.
   c) if $a, b \in R$, and b is not a zero-divisor, then $((ab) : (b)) = (a)$.

8. The radical of an ideal I is $\sqrt{I} = \{r \in R;\, r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}$. An ideal I is called a radical ideal, if $I = \sqrt{I}$. Prove that $\sqrt{I}$ is indeed an ideal. Then check the following formulas, where $I, J \subseteq R$ are ideals: $I \subseteq \sqrt{I}$, $\sqrt{\sqrt{I}} = \sqrt{I}$, $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$, $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$.