Höhere Algebra (Säule II)

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It suffices to solve 6 out of the following 10. Feel free to solve more in order to collect some extra points!

1. Let F be the splitting field of $X^4 - 4X^2 + 2$ over $\mathbb{Q}$. Show that $F = \mathbb{Q}(\sqrt{2+\sqrt{2}})$. Describe the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$, the subgroups of G, the subfields of F, and the bijection given by the main theorem of Galois theory. (Hint: $(2+\sqrt{2})(2-\sqrt{2}) = 2$.)

2. Show that $\mathbb{C}(X) \supseteq \mathbb{C}(X^6)$ is a Galois extension. Describe the Galois group $G = \mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(X^6))$, the subgroups of G, the intermediate fields, and the bijection given by the main theorem of Galois theory.

3. Let $F \supseteq K$ be a finite Galois extension. Let $f \in K[X]$ be an irreducible polynomial. Let $\alpha, \beta \in F$ be two zeros of f. Show that there is a $\sigma \in \mathrm{Gal}(F/K)$ such that $\sigma(\alpha) = \beta$.

4. Let $f \in \mathbb{Q}[X]$ be an irreducible polynomial. Show that if f has a zero in both $\mathbb{R}$ and $\mathbb{C} \setminus \mathbb{R}$, then the Galois group of f (that is, the Galois group of the splitting field of f over $\mathbb{Q}$) is non-abelian. (Hint: Complex conjugation and Ex. 3. can be useful.)

5. Let R be an integrally closed integral domain, K the quotient field of R, and $F \supseteq K$ a field extension. Let $\alpha \in F$ be an algebraic element over K, and let $f \in K[X]$ be the (monic) minimal polynomial of α over K. Show that α is integral over R if and only if $f \in R[X]$.

6. Let $d \in \mathbb{Q}\setminus\mathbb{Z}$. Show that $\mathbb{Z}[\sqrt{d}]$ is not a finitely generated abelian group.

7. Let $F \supseteq K$ be a Galois extension and $R \subseteq F$ be a subring. Show that if $\tau(R) = R$ for every $\tau \in \mathrm{Gal}(F/K)$, then R is integral over $R \cap K$.

8. Let $d \in \mathbb{Z} \setminus \{0, 1\}$ be a squarefree integer (so $p^2 \nmid d$ for all primes p). Which elements of the field $\mathbb{Q}(\sqrt{d})$ are integral over $\mathbb{Z}$? (Hint: $\mathbb{Q}(\sqrt{d}) = \mathbb{Q} + \mathbb{Q}\sqrt{d}$.)

9. Let $B \supseteq A$ be an integral extension. Show that B is a field if and only if A is a field.

10. Let K be a field and let $(A_i)_{i\in I}$ be a set of integrally closed integral domains in K whose field of fractions equals K. Show that $A := \bigcap_{i\in I} A_i$ is an integrally closed ring.