Höhere Algebra (Säule II)

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In the first four exercises, F denotes the splitting field of $f = X^4 - 2$ over $\mathbb{Q}$, and $G = \mathrm{Gal}(F/\mathbb{Q})$.

1. Show that $F = \mathbb{Q}(\sqrt[4]{2}, i)$ and $|G| = [F : \mathbb{Q}] = 8$. Find several examples of permutations on the zeros of f that do not extend to $\mathbb{Q}$-automorphisms of F. How many such permutations are there?

2. Show that G is generated by unique elements $\sigma, \tau \in G$ such that $\sigma(\sqrt[4]{2}) = i\sqrt[4]{2}$, $\sigma(i) = i$, $\tau(\sqrt[4]{2}) = \sqrt[4]{2}$, $\tau(i) = -i$ and that they satisfy the following relations: $\sigma^4 = 1$, $\tau^2 = 1$, $\tau\sigma = \sigma^{-1}\tau$. (This 8 element group is called the dihedral group D_4.) Conclude that $G = \{\sigma^a\tau^b;\ a \in \mathbb{Z}/4\mathbb{Z},\ b \in \mathbb{Z}/2\mathbb{Z}\}$, and the multiplication is given by the formula $(\sigma^a\tau^b)(\sigma^c\tau^d) = \sigma^{a+(-1)^b c}\tau^{b+d}$.

3. Determine the subgroups of G (or equivalently, of D_4), and draw the lattice of these subgroups. (Hint: there are 10 subgroups in total. You can use the notation $\langle \dots \rangle$ to denote subgroups: e.g. $\langle \sigma^2, \tau \rangle$ is the subgroup of G generated by σ^2 and τ.)

4. For each subgroup H of G, find the corresponding fixed field F^H. (Hint: You may use that conjugate subgroups correspond to conjugate fields. Another trick: if $\theta \in G$ and $\theta^2 = 1$, then $x + \theta(x)$ is fixed by θ for every $x \in F$.)

It suffices to solve 2 out of the following 4. Feel free to solve more in order to collect some extra points!

5. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $G = \mathrm{Gal}(K/\mathbb{Q})$. Describe G, the subgroups of G, the subfields of K, and the bijection given by the main theorem of Galois theory.

6. Let $F \supseteq E \supseteq K$ with $F \supseteq K$ Galois and $[F : K] = n < \infty$. Prove that the order of $\mathrm{Gal}(F/E)$ is equal to $n/[E : K]$.

7. Let $L \supseteq K$ be a finite separable field extension. Show that there are only finitely many intermediate fields between L and K. (Hint: Take the smallest normal extension containing L in the algebraic closure.)

8. Let $\Omega \supseteq K$ be a field extension, and let L_1, L_2 be intermediate fields such that $L_1 \supseteq K$ and $L_2 \supseteq K$ are finite Galois extensions, and $L_1 \cap L_2 = K$. Show that $L_1L_2 \supseteq K$ is also a finite Galois extension, and $\mathrm{Gal}(L_1L_2/K) \cong \mathrm{Gal}(L_1/K) \times \mathrm{Gal}(L_2/K)$.