Höhere Algebra (Säule II)

Übung, LVA 405.451

C. Fuchs, R. Paulin

10. Übungsblatt, WS 2013/14 — 10.01.2014

It suffices to solve 6 out of 8. Feel free to solve more in order to collect some extra points!

1. Show that every field extension of degree 2 is normal.

2. a) Show that $\mathbb{Q}(\zeta_p) \supseteq \mathbb{Q}$ is a normal extension, where p is a prime and $\zeta_p = e^{2\pi i/p} \in \mathbb{C}$.

   b) More generally, show that if $L \supseteq K$ is a field extension, and $\zeta^n = 1$ for some $\zeta \in L$ and $n \in \mathbb{Z}_{\geq 1}$, then $K(\zeta) \supseteq K$ is a normal extension.

3. Let K be a field such that $X^n - 1$ splits into linear factors in $K[X]$. Let $K(\alpha) \supseteq K$ be a field extension where $\alpha^n \in K$. Show that $K(\alpha) \supseteq K$ is a normal extension.

4. Let $\alpha \in \mathbb{C}$ be a zero of $f(X) = X^3 - 3X + 1$. Show that $\mathbb{Q}(\alpha) \supseteq \mathbb{Q}$ is a normal extension. (Hint: Check that $\alpha^2 - 2$ is also a zero of f.)

5. Let $F \supseteq K$ is a field extension, and let $F \supseteq L_i \supseteq K$ be intermediate fields ($i \in \{1, 2\}$). Show that if $L_1 \supseteq K$ and $L_2 \supseteq K$ are normal, then $L_1 L_2 \supseteq K$ and $L_1 \cap L_2 \supseteq K$ are also normal.

6. Find an example of fields $F \supseteq L \supseteq K$ such that $F \supseteq L$ and $L \supseteq K$ are normal, but $F \supseteq K$ is not normal.

7. Let K be a field and $f(X) \in K[X]$ a polynomial of degree n. Let L be the splitting field of f. Show that $[L : K] \mid n!$. (Hint: Prove by induction.)

8. Show that if $F \supseteq E \supseteq K$ are fields such that $F \supseteq K$ is normal, then every automorphism $\sigma \in \mathrm{Aut}_K(E)$ can be extended to an automorphism $\widetilde{\sigma} \in \mathrm{Aut}_K(F)$ (so $\widetilde{\sigma}|_E = \sigma$).