Höhere Algebra (Säule II)

Übung, LVA 405.451
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1. Prove the following assertions in a unitary ring R:
   a) The unit element 1 is unique.
   b) If $1 = 0$, then $R = \{0\}$.
   c) If $n \in \mathbb{N}$ and $n1 = 0$, then $na = 0$ for all $a \in R$.

2. Prove that every additive abelian group G is a ring without unit if we define $ab = 0$ for all $a, b \in G$.

3. Let R be a ring without unit. As both $\mathbb{Z}$ and R are additive groups, so is their product $R^* = \mathbb{Z} \times R$. Define a multiplication on R^* by

$$(m, r)(n, s) = (mn, ms + nr + rs),$$

   where $ms = 0$ if $m = 0$, ms is the sum of $s \in R$ with itself m times if $m > 0$, and ms is the sum of $-s$ with itself $|m|$ times if $m < 0$. Prove that R^* is a unitary ring (one says that R^* arises from R by adjoining a unit).

4. If R is a nonzero commutative ring, let $\mathcal{F}(R)$ be the set of all functions from R to R with pointwise addition and pointwise multiplication. Show that $\mathcal{F}(R)$ is a commutative ring, which is not a domain, and that $\mathcal{F}(\mathbb{Z}/2\mathbb{Z})$ has exactly four elements and satisfies $f + f = 0$ for all $f \in \mathcal{F}(\mathbb{Z}/2\mathbb{Z})$.

5. Prove that the only subring of $\mathbb{Z}$ is $\mathbb{Z}$ itself.

6. A ring R is called a boolean ring if $\forall a \in R$: $a^2 = a$. Prove that a boolean ring is necessarily commutative. Show for $|R| < \infty$ that R is isomorphic to a direct product of copies of $\mathbb{Z}/2\mathbb{Z}$.

7. Prove that every ring R with unit can be embedded (i.e. there is an injective homomorphism) into the endomorphism ring of an additive abelian group.

8. Determine a system of pairwise orthogonal idempotent elements in the group ring $\mathbb{R}[\mathbb{Z}/2\mathbb{Z}]$. Show that this group ring is isomorphic to $\mathbb{R} \times \mathbb{R}$.