Exercise 7/8.

Let $\mathbb{Q}(x)/\mathbb{Q}$ be a transcendental extension, and let $y = \frac{f(x)}{g(x)} \in \mathbb{Q}(x)$, where $f, g \in \mathbb{Q}[x]$, $g \neq 0$, $(f, g) = 1$, and $\deg f > 0$ or $\deg g > 0$. Show that $[\mathbb{Q}(x) : \mathbb{Q}(y)] = \max(\deg f, \deg g)$.

Hint: First step: We need that $\mathbb{Q}[y]$ is the polynomial ring, so we need to prove that y is transcendental over $\mathbb{Q}$. To do this, first recall how we can find rational zeros of polynomials in $\mathbb{Z}[X]$, then note that $\mathbb{Z}$ could be replaced by any UFD, and apply this for $\frac{f}{g}$.
Second step: Using the equation $y = \frac{f(x)}{g(x)}$, try find a nonzero polynomial $F \in (\mathbb{Q}[y])[T]$ such that $F(x) = 0$. This proves that x is algebraic over $\mathbb{Q}(y)$.
Third step: Finally, we need to prove that F is irreducible (for the right choice of F), giving the degree $[\mathbb{Q}(x) : \mathbb{Q}(y)]$. Here one can use that $\mathbb{Q}[y][T] = \mathbb{Q}[T][y]$.