THOMAS' FAMILY OF THUE EQUATIONS OVER FUNCTION FIELDS

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Abstract. We consider a function field analogue of Thomas' family [14] of Thue equations

$$X^3 - (\lambda - 1)X^2Y - (\lambda + 2)XY^2 - Y^3 = \xi,$$

where the solutions X, Y come from the ring $\mathbb{C}[T]$, the parameter $\lambda \in \mathbb{C}[T]$ is some non-constant polynomial and $0 \neq \xi \in \mathbb{C}$. In this paper we completely solve this family.

1. Introduction and Main Result

The history of Thue equations, these are diophantine equations $F(X, Y) = m$ where $F \in \mathbb{Z}[X, Y]$ is a binary irreducible form of degree at least 3 and m is a non zero integer, in particular the study of families of Thue equations dates back to Thue himself. In his paper [15] he proved that the Thue equation

$$(a + 1)X^n - aY^n = 1$$

has only the solution $x = y = 1$, if a is suitable large in relation to the prime $n \geq 3$. In more recent time Thomas [14] considered the family

$$X^3 - (\lambda - 1)X^2Y - (\lambda + 2)XY^2 - Y^3 = \pm 1 \quad (\lambda \in \mathbb{Z}). \tag{1}$$

This was the first time that a family of Thue equations with positive discriminant has been investigated. Thomas [14] and Mignotte [11] solved this equation completely. It has the solutions $\pm(1, 0), \pm(0, 1), \pm(1, -1)$ for all $\lambda \in \mathbb{Z}$ and some further solutions for $\lambda = 0, 1, 3$. Since then several authors considered many families of Thue equations up to degree 8 (cf. [6]). A survey on the subject can be found in [4].

All the families stated above were considered over the ring of rational integers. Heuberger, Pethő and Tichy solved the first time a family of relative Thue equations in [5]. In particular they studied Thomas' family again

$$X^3 - (\lambda - 1)X^2Y - (\lambda + 2)XY^2 - Y^3 = \mu, \tag{2}$$

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where the solutions x, y, the parameter λ and the root of unity μ now are all (algebraic) integers in the same imaginary quadratic number field k. They proved that (2) has only trivial solutions (i.e. with $|x|, |y| \leq 1$), if $|\lambda|$ is large enough or if the discriminant of the quadratic number field is large enough or if $\Re\lambda = -1/2$ (there are a few more solutions in this case which are explicitly listed).

Thue equations were not only considered over number fields but also over function fields starting with the paper of Gill [3]. In the next 50 years several authors as Schmidt (cf. [13]) and Mason (cf. [9], resp. [10]) considered the problem to determine effectively all solutions of a given Thue equation over some function field. In contrast to the number field case Thue equations over function fields may have infinitely many solutions. Recently, Lettl [8] could prove criteria for which a given Thue equation has only finitely many solutions. In all these investigations no families of Thue equations were considered. Recently, the authors could solve the family

$$X(X-Y)(X-(T+\xi)Y)+Y^3 = 1+\xi T(1-T)$$

over $\mathbb{C}[T]$ for all $\xi \in \mathbb{C}$ (cf. [2]). This was the first time that a family of Thue equations over a function field was solved.

In this paper we will go a step further and consider a family, where the parameter itself is a polynomial. In particular we revisit Thomas' family and we completely solve the equation

$$X^3 - (\lambda-1)X^2Y - (\lambda+2)XY^2 - Y^3 = \xi, \tag{3}$$

where $\lambda \in \mathbb{C}[T]$ is a non-constant polynomial and $\xi \in \mathbb{C}^\times$ (which denotes as usually the unit group and in this case is just $\mathbb{C}\backslash\{0\}$). This family can be seen as a function field analogue of the Thue equations (1) and (2), respectively. In all three cases the solutions x, y and the parameter λ come from the same commutative ring R, namely $\mathbb{Z}, \mathfrak{o}_k$ and $\mathbb{C}[T]$, respectively, and the right hand side is a unit in R (we denote by $\mathfrak{o}_k$ the maximal order of some number field k).

The aim of this paper is to prove the following main theorem:

Theorem 1. *If $(x,y) \in \mathbb{C}[T] \times \mathbb{C}[T]$ is a solution of the Thue equation*

$$X^3 - (\lambda-1)X^2Y - (\lambda+2)XY^2 - Y^3 = \xi,$$

with non-constant $\lambda \in \mathbb{C}[T]$ and $\xi \in \mathbb{C}^\times$, then

$$(x,y) \in \mathcal{L} := \left\{(\zeta,0),(-\zeta,\zeta),(0,-\zeta)\ :\ \zeta^3 = \xi\right\}.$$

Observe that the result contains all the (infinite) polynomial families of solutions found by Thomas [14] in his original paper and by Heuberger, Pethő and Tichy in [5] (see Table 1, page 438). So, our main theorem gives once again all the polynomial solutions in these cases and therefore we proved one part of what Thomas calls *stable growth*, which means that a family of Thue equations has only finitely many polynomial solutions plus finitely many sporadic ones (which means solutions for certain values of the parameter).

The rest of the paper is organized as follows. Section 2 provides a collection of useful facts on function fields and auxiliary results that we need to prove Theorem 1. In Section 3 we compute ramification indices and the genus of the function field $\mathbb{C}(T)(\alpha)$, where α is a root of $X^3-(\lambda-1)X^2-(\lambda+2)X-1$. These computations yield an upper bound for $H_K(x-\alpha y)$ (cf. Section 4). The structure of the relevant unit group is studied in Section 5. Finally, we prove our main theorem in Section 6.

2. Preliminaries and auxiliary Results

Let us first remind the ABC-Theorem for function fields (see e.g. [12, Theorem 7.17]).

Proposition 1 (ABC-Theorem)**.** *Let K be a function field of characteristic 0 and genus g_K. Let $u, v \in K^\times$ satisfying $u+v=1$ and put $A=(u)_0$, $B=(v)_0$ and $C=(u)_\infty=(v)_\infty$, where $(\cdot)_0$ denotes the zero divisor and $(\cdot)_\infty$ denotes the polar divisor. Then*

$$\deg A = \deg B = \deg C \leq \max\left(0, 2g_K - 2 + \sum_{P\in \operatorname{Supp}(A+B+C)} \deg_K P\right).$$

If the constant field k is algebraically closed and of characteristic 0, Mason [10, chapter 1, Lemma 2] proved following special case.

Corollary 1. *Let $H(f) := -\sum_{v\in M_K} \min(0, v(f))$ denote the height of an element $f\in K$ and let $\gamma_1,\gamma_2,\gamma_3 \in K$ with $\gamma_1+\gamma_2+\gamma_3=0$. Let $\mathcal{V}$ be a finite set of valuations such that for all $v\notin\mathcal{V}$ we have $v(\gamma_1)=v(\gamma_2)=v(\gamma_3)$, then*

$$H(\gamma_1/\gamma_2) \leq \max(0, 2g-2+|\mathcal{V}|).$$

Here we denote the set of all valuations in K by M_K. Usually, M_K denotes the set of places of some field K. Since in the function field case valuations and places are one-to-one we use this notation. It is rather easy to deduce Corollary 1 from Proposition 1 (cf. [2]).

If $F(X,Y)=m$ is a Thue equation over the integral closure $\mathcal{O}_L$ of $k[T]$ in some function field $L/k(T)$, then Mason [9] could prove an effective bound for the height of solutions (x,y) to $F(X,Y)=m$ by using his fundamental inequality presented in Corollary 1. Observe that the integral closure of $k[T]$ consists of all elements in L that may only have negative valuations above ∞. Let us remind Mason's bound on the height of the solutions.

Proposition 2 (Mason)**.** *Let*

$$F(X,Y) := (X-\alpha_1 Y)\cdots(X-\alpha_d Y) = m$$

be a Thue equation over the integral closure $\mathcal{O}_L$ of $k[T]$ of some function field $L/k(T)$ with algebraic closed constant field and characteristic 0. Then *all solutions* $(x,y)\in\mathcal{O}_L^2$ *satisfy*

$$H(x,y)\leq 8H+2g_L+r-1,$$

where g_L *is the genus of* L*,* r *is the number of infinite valuations and* H *denotes the height of the polynomial* $(X-\alpha_1)\cdots(X-\alpha_d)/m$*.*

In order to compute the genus of some function field we will use the Riemann-Hurwitz Theorem (see e.g. [12, Theorem 7.16])

Proposition 3 (Riemann-Hurwitz)**.** *Let* L/K *be an extension of function fields of characteristic* 0 *and let* g_K *and* g_L *be the genus of* K *and* L*, respectively, then*

$$2g_L-2=[L:K](2g_K-2)+\sum_{w\in M_L} e_w-1, \tag{4}$$

where M_L *is the set of valuations of* L *and* e_w *denotes the ramification index of* w *in the extension* L/K*.*

Let us fix for the rest of the paper some notations.

$$F_\lambda(X,Y):=X^3-(\lambda-1)X^2Y-(\lambda+2)XY^2-Y^3, \tag{5}$$

$$f_\lambda(X):=F_\lambda(X,1):=X^3-(\lambda-1)X^2-(\lambda+2)X-1, \tag{6}$$

where $\lambda\in\mathbb{C}[T]$ is some non-constant polynomial. We denote the degree of λ by $\mathfrak{a}:=\deg\lambda>0$. Let $\alpha:=\alpha_1$ be a root of f_λ and let α_2 and α_3 be the conjugates of α. Then $K:=\mathbb{C}(T)(\alpha)$ is a finite extension of the function field $\mathbb{C}(T)$. Observe that we have the identities

$$F_\lambda(Y,-(X+Y))=F_\lambda(-(X+Y),X)=F_\lambda(X,Y)\quad\text{and}$$
$$f_\lambda\left(-1-\frac{1}{X}\right)=\frac{f_\lambda(X)}{X^3}.$$

Therefore,

$$\alpha_2=-1-\frac{1}{\alpha}\quad\text{and}\quad\alpha_3=-1-\frac{1}{\alpha_2}=-\frac{1}{\alpha+1}.$$

Observe that f_λ is irreducible over $\mathbb{C}(T)$, since otherwise one of the roots would be in $\mathbb{C}[T]$ (because all the root are integral over $\mathbb{C}[T]$), which contradicts the fact that the constant term of f_λ is -1, unless this root is in $\mathbb{C}\backslash\{0\}$. But this implies by the above formulas that all roots are in $\mathbb{C}$, which is clearly not possible. Consequently, K is the splitting field of f_λ and hence a Galois extension of $\mathbb{C}(T)$. Furthermore, the Galois group is the cyclic group with three elements. We collect these facts in the following

Lemma 1. *The extension* $K/\mathbb{C}(T)$ *is a Galois extension. In particular the Galois group* G *of* $K/\mathbb{C}(T)$ *is isomorphic to the cyclic group* C_3 *with three elements.*

Further, we denote by $(x,y)\in\mathbb{C}[T]\times\mathbb{C}[T]$ some solution to $F_\lambda(X,Y)=\xi$ with $\xi\in\mathbb{C}^\times$ and we define

$$\alpha_{i,j}:=\alpha_i-\alpha_j,\qquad \beta_i:=x-\alpha_i y,\qquad \gamma_{i,j,k}:=\beta_i\alpha_{j,k},$$

and we write $\beta:=\beta_1=x-\alpha y$. The polynomial F_λ may be expressed as $F_\lambda(X,Y)=\mathcal{N}_{K/\mathbb{C}(T)}(X-\alpha Y)$, where $\mathcal{N}_{K/\mathbb{C}(T)}$ denotes the norm from K to $\mathbb{C}(T)$. From this norm notation we deduce $\beta_i\in\mathbb{C}[T][\alpha]^\times$.

Note that since $\mathbb{C}$ is algebraically closed all discrete valuation rings with quotient field $\mathbb{C}(T)$ are isomorphic to $O_a:=\{f(T)/g(T)\,:\,f(T),g(T)\in\mathbb{C}[T],g(a)\neq 0\}$ for some $a\in\mathbb{C}$ or $O_\infty:=\{f(T)/g(T)\,:\,f(T),g(T)\in\mathbb{C}[T],\deg f\leq\deg g\}$. Valuations corresponding to O_a are denoted by v_a (finite valuations) and the valuation corresponding to O_∞ is denoted by v_∞ (infinite valuation). To simplify notations we will use the following definition.

Definition 1. *Let $L/\mathbb{C}(T)$ be an extension of degree d, $f\in L$ and for each $a\in\mathbb{C}\cup\{\infty\}$ let us fix a d-tuple $(w_1,\dots,w_d)$ of valuations in L with $w_i|v_a$ for $i\in\{1,\dots,d\}$, where the valuations are used with multiplicity, i.e. a valuation with ramification index e is written down e times. Let us define*

$$(\cdot)_a:L\to\mathbb{Z}^d,\qquad f\mapsto(f)_a:=(w_1(f),\dots,w_d(f)).$$

In the next lemma we calculate all valuations of the quantities $\alpha_1,\alpha_2,\alpha_3$, respectively, in the function field $K/\mathbb{C}(T)$.

Lemma 2. *We have*

$$(\alpha_1)_\infty=(\mathfrak{a},0,-\mathfrak{a}),\qquad(\alpha_2)_\infty=(0,-\mathfrak{a},\mathfrak{a}),\qquad(\alpha_3)_\infty=(-\mathfrak{a},\mathfrak{a},0)$$

and $(\alpha_i)_a=(0,0,0)$ for $a\in\mathbb{C}$ and $i=1,2,3$, where $\mathfrak{a}=\deg\lambda$.

Proof: Let us first consider the infinite valuations. We compute α_i as formal Laurent series in λ and obtain

$$\begin{aligned}\alpha_1=&\lambda+1+\frac{1}{\lambda}-\frac{3}{\lambda^2}+\cdots=\lambda_{\mathfrak{a}}T^{\mathfrak{a}}+\cdots,\\ \alpha_2=&-1-\frac{1}{\lambda}+\frac{1}{\lambda^2}+\cdots=-1-\cdots,\\ \alpha_3=&-\frac{1}{\lambda}+\frac{2}{\lambda^2}+\cdots=-\frac{1}{\lambda_{\mathfrak{a}}}T^{-\mathfrak{a}}+\cdots,\end{aligned}$$

where $\lambda_{\mathfrak{a}}$ denotes the leading term of λ. From the expansion above we obtain the first part of the lemma by using a suitable triple of the infinite valuations. The second part is immediate, since α_1,α_2 and α_3 are integer units in K. ■

3. Ramification

In this section we compute, which places are ramified in the extension $K/\mathbb{C}(T)$. Since $K/\mathbb{C}(T)$ is Galois, the ramification indices e_w may only take the value 1 (unramified) or 3 (ramified). Hence,it suffices to know how often ramification occurs in order to compute the genus g_K of K using Proposition 3. Let us first prove

Lemma 3. *If $w \in M_K$ is ramified then $w|v_a$, where $a \in \mathbb{C}$ such that a is a root of $\delta := \lambda^2 + \lambda + 7 \in \mathbb{C}[T]$.*

Proof: Assume K is not ramified at a valuation that lies over v_a with $a \neq \infty$. By Puiseux's Theorem [1, 7] there exists a formal Power series

$$\alpha(T) := \sum_{n=0}^{\infty} a_n (T-a)^n$$

such that $f_\lambda(\alpha(T)) = 0$. On the other hand it is well known that given an equation $f(X,T) = 0$ with f holomorphic and $\left.\frac{\partial f}{\partial X}\right|_{T=a} \neq 0$, then there exists a holomorphic function $X(T) = \sum_{n=0}^{\infty} a_n(T-a)^n$ in an open neighborhood $U \subset \mathbb{C}$ of a, such that $f(X(T),T) = 0$. We conclude that ramification over v_a may only occur if

$$\begin{aligned} \left.\frac{\partial f_\lambda}{\partial X}\right|_{T=a} &= 3X^2 - 2X(\lambda(a)-1) - (\lambda(a)+2) = 0 \quad \text{and} \\ f_\lambda(X)|_{T=a} &= X^3 - X^2(\lambda(a)-1) + X(\lambda(a)+2) - 1 = 0 \end{aligned}$$

or equivalently $f_\lambda|_{T=a}$ has a multiple zero. This is $\delta(a) := \delta|_{T=a} = \lambda(a)^2 + \lambda(a) + 7 = 0$.

We are left to prove that v_∞ is not ramified. Since the infinite valuations of K have different values α by Lemma 2, these are different and thus unramified over $\mathbb{C}(T)$. ■

Corollary 2. *If r is the number of ramified valuations in $K/\mathbb{C}(T)$ then $g_K = r - 2 \leq 2\mathfrak{a} - 2$.*

Proof: For the proof we utilize Proposition 3. We already know that if $w \in M_K$ is ramified then $e_w = 3$. Moreover, the genus of $\mathbb{C}(T)$ is zero. With this data we obtain

$$2g_K - 2 = 3(0-2) + (3-1)r = 2r - 6,$$

hence, $g_K = r - 2$. By Lemma 3 we therefore have $g_K = r - 2 \leq 2\mathfrak{a} - 2$. ■

4. Upper bound for the height of β

In order to solve equation (3) we have to compute all values of $\beta = x - \alpha y$ that solve the norm equation

$$\mathcal{N}_{K/\mathbb{C}(T)}(\beta) = \xi \quad (\xi \in \mathbb{C}^\times). \tag{7}$$

Note that β is a unit of $\mathbb{C}[T][\alpha]$. A first step is to bound the height of β, which will be done in this section.

Lemma 4. *We have*
$$H_K\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) \le 6\mathfrak{a} - 3.$$

Proof: It is well known that
$$\gamma_{1,2,3} + \gamma_{2,3,1} + \gamma_{3,1,2} = 0$$
(Siegel's Identity), hence by Corollary 1 we have
$$H_K\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) \le \max(0, 2g_K - 2 + |\mathcal{V}|).$$
Let r be the number of ramified places, then $g_K = r - 2$ (cf. Corollary 2). Since $\gamma_{i,j,k}$ with pairwise distinct $i, j, k \in \{1,2,3\}$ may only have non zero valuations at places w with $w|v_a$, where $a \in \{k \in \mathbb{C} : \delta(k) = 0\} \cup \{\infty\}$, we have $|\mathcal{V}| \le r + 3(2\mathfrak{a} - r) + 3$. This yields
$$H_K\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) \le 2r - 4 - 2 + r + 3(2\mathfrak{a} - r) + 3 = 6\mathfrak{a} - 3.$$
■

Next, we want to compute an upper bound for $H_K(\beta_1/\beta_2)$. Let us denote by
$$H_a(\alpha) := -\sum_{w|v_a} \min(0, w(\alpha)), \quad a \in \mathbb{C} \cup \{\infty\}$$
the local height. Obviously, we have
$$H_K(\alpha) = \sum_{a \in \mathbb{C} \cup \{\infty\}} H_a(\alpha). \tag{8}$$
From equation (8) one computes
$$\begin{aligned} H_K\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) &= \sum_{\delta(a)=0} H_a\left(\frac{\alpha_{2,3}}{\alpha_{3,1}}\right) + H_\infty\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) \\ &\ge H_\infty\left(\frac{\beta_1}{\beta_2} \cdot \frac{\alpha_{2,3}}{\alpha_{3,1}}\right) \\ &\ge H_\infty\left(\frac{\beta_1}{\beta_2}\right) - H_\infty\left(\frac{\alpha_{3,1}}{\alpha_{2,3}}\right). \end{aligned} \tag{9}$$
By Lemma 2 we obtain $\left(\frac{\alpha_{3,1}}{\alpha_{2,3}}\right)_\infty = (0, \mathfrak{a}, -\mathfrak{a})$, hence
$$H_K\left(\frac{\beta_1}{\beta_2}\right) = H_\infty\left(\frac{\beta_1}{\beta_2}\right) \le H_K\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) + H_\infty\left(\frac{\alpha_{3,1}}{\alpha_{2,3}}\right) \le 7\mathfrak{a} - 3.$$

Let us now compute $H_K(\beta)$. Since $\beta_i \in \mathbb{C}[T][\alpha]^\times$, with $i = 1, 2, 3$, we only have to consider infinite valuations. After a suitable permutation of valuations we may assume
$$(\beta_1)_\infty = (-b_1, -b_2, -b_3)$$

where $b_1 \geq b_2 \geq b_3$. Therefore, the indices of the conjugates of β_1 are fixed. By Lemma 1 we either have (case 1)

$$(\beta_2)_\infty = (-b_3, -b_1, -b_2)$$
$$(\beta_3)_\infty = (-b_2, -b_3, -b_1)$$

or (case 2)

$$(\beta_2)_\infty = (-b_2, -b_3, -b_1)$$
$$(\beta_3)_\infty = (-b_3, -b_1, -b_2)$$

which implies that

$$(\beta_1/\beta_2)_\infty = \begin{cases} (b_3 - b_1, b_1 - b_2, b_2 - b_3) & \text{(case 1)}, \\ (b_2 - b_1, b_3 - b_2, b_1 - b_3) & \text{(case 2)}. \end{cases}$$

Consequently, we get

$$H_K(\beta_1/\beta_2) = \left\{ \begin{array}{l} b_1 - b_3 \\ b_1 - b_2 - b_3 + b_2 = b_1 - b_3 \end{array} \right\} = b_1 - b_3.$$

Since $H_K(\beta) = H_K(\beta^{-1})$ we may also assume $b_1 \geq b_2 \geq 0$. From $b_1 + b_2 + b_3 = 0$ we deduce $-b_3 = b_1 + b_2 \leq 2b_1$, hence $\frac{-b_3}{2} \leq b_1$. This yields

$$\frac{3}{2} H_K(\beta) = -\frac{3}{2} b_3 \leq b_1 - b_3 = H_K(\beta_1/\beta_2) \leq 7\mathfrak{a} - 3,$$

hence following Proposition:

Proposition 4. *We have* $H_K(\beta) \leq \frac{14}{3}\mathfrak{a} - 2$.

We remark that by using Proposition 2 directly we get the weaker result that if (x, y) is a solution to (3) then

$$\max(\deg x, \deg y) \leq \frac{28\mathfrak{a} - 2}{3}.$$

This can be obtained just by computing $H = H_K(f_\lambda)$, since K is the splitting field of f_λ:

$$H = \max\left(H_K(1), H_K(\lambda - 1), H_K(\lambda + 2), H_K(-1)\right) = 3\mathfrak{a},$$

where H_K denotes the height with respect to the function field K. Now $r = 3$ by Lemma 3 and $g_K \leq 2\mathfrak{a} - 2$ by Corollary 2, hence,

$$\max(H_K(x), H_K(y)) \leq 24\mathfrak{a} + 4\mathfrak{a} - 4 + 3 - 1 = 28\mathfrak{a} - 2.$$

Since $x, y \in \mathbb{C}[T]$ we have $3 \deg x = H_K(x)$, $3 \deg y = H_K(y)$, respectively, and therefore the claimed bound.

5. Unit structure

Since $\beta \in \mathbb{C}[T][\alpha]^\times$ we have to investigate this unit group. The aim of this section is to prove the following precise description of this group

Proposition 5. *The units α_1 and α_2 form a fundamental system, so $\mathbb{C}[T][\alpha]^\times = \mathbb{C}\langle\alpha_1, \alpha_2\rangle$, i.e. for every $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ we have*

$$\varepsilon = \eta\alpha_1^x\alpha_2^y,$$

where $\eta \in \mathbb{C}$ and x, y are integers.

In order to prove Proposition 5 we consider first the following Lemma.

Lemma 5. *Let $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ then either $H_K(\varepsilon) \geq \mathfrak{a}$ or $\varepsilon \in \mathbb{C}^\times$.*

Proof: Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ elements of K with $(\varepsilon_1)_\infty = (a_1, b_1, c_1), (\varepsilon_2)_\infty = (a_2, b_2, c_2)$ and $(\varepsilon_3)_\infty = (a_3, b_3, c_3)$. We will write by abuse of language $(a_1, b_1, c_1) \circ (a_2, b_2, c_2) = (a_3, b_3, c_3)$, if $\varepsilon_1 \circ \varepsilon_2 = \varepsilon_3$, where $\circ = +, -, \cdot$. A computation (if $\circ = +, -$ and e.g. $a_1 = a_2$) may not be unique in this vector notation. In this case we place a ? at that component.

Let $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ and $(\varepsilon)_\infty = (e_1, e_2, e_3)$. Since $H_K(\varepsilon) = H_K(\varepsilon_i) = H_K(\varepsilon_i^{-1})$ with $i = 1, 2, 3$ and where ε_i are the conjugates of ε, we may assume $e_1, e_3 \geq 0$ and $e_2 < 0$. We may exclude $e_2 = 0$, since otherwise ε would be constant. We have

$$\varepsilon_i = h_0 + h_1\alpha_i + h_2\alpha_i^2 \quad (1 \leq i \leq 3), \tag{10}$$

with $h_0, h_1, h_2 \in \mathbb{C}[T]$. Solving this linear system by Cramer's rule one obtains

(11)

$$h_0 = \frac{\varepsilon_1\alpha_2\alpha_3(\alpha_3 - \alpha_2) + \varepsilon_2\alpha_3\alpha_1(\alpha_1 - \alpha_3) + \varepsilon_3\alpha_1\alpha_2(\alpha_2 - \alpha_1)}{\delta},$$

$$h_1 = \frac{\varepsilon_1(\alpha_2 + \alpha_3)(\alpha_2 - \alpha_3) + \varepsilon_2(\alpha_3 + \alpha_1)(\alpha_3 - \alpha_1) + \varepsilon_3(\alpha_1 + \alpha_2)(\alpha_1 - \alpha_2)}{\delta},$$

$$h_2 = \frac{\varepsilon_1(\alpha_3 - \alpha_2) + \varepsilon_2(\alpha_1 - \alpha_3) + \varepsilon_3(\alpha_2 - \alpha_1)}{\delta},$$

where

$$\delta = \det(\alpha_j^{i-1})_{1 \leq i,j \leq 3} = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1) = \lambda^2 + \lambda + 7 \in \mathbb{C}[T]$$

is the discriminant of f_λ.

Let us assume $h_0 \neq 0$. We want to compute $\deg h_0$. Using the vector notation we obtain

$$(\alpha_2\alpha_3(\alpha_3 - \alpha_2))_\infty = (0, -\mathfrak{a}, \mathfrak{a})(-\mathfrak{a}, \mathfrak{a}, 0)((-\mathfrak{a}, \mathfrak{a}, 0) - (0, -\mathfrak{a}, \mathfrak{a})) = (-2\mathfrak{a}, -\mathfrak{a}, \mathfrak{a})$$

and

$$(\delta)_\infty = (-2\mathfrak{a}, -2\mathfrak{a}, -2\mathfrak{a}),$$

hence

$$(h_0)_\infty = (e_1, \mathfrak{a} + e_2, 3\mathfrak{a} + e_3) + \cdots,$$

where the other summands are the conjugates of the first summand. We have $0 \geq -\deg h_0 \geq \min(e_1, e_2+\mathfrak{a})$. But $\min(e_1, e_2+\mathfrak{a}) \leq 0$, if either $e_1 = 0$ and $H_K(\varepsilon) = -e_2 < \mathfrak{a}$ or $H_K(\varepsilon) = -e_2 \geq \mathfrak{a}$. So either the Lemma is true or $h_0 \in \mathbb{C}$.

If $h_0 \in \mathbb{C}$ one has $H_K(\varepsilon) = H_K(h_1\alpha + h_2\alpha^2)$. Therefore, without loss of generality we may assume that $h_0 = 0$. Let us assume now $h_1 \neq 0$, $h_2 \neq 0$ and furthermore, $\deg h_1 = \mathfrak{h}_1$ and $\deg h_2 = \mathfrak{h}_2$. Since $\varepsilon = \alpha(h_1 + h_2\alpha)$, we have

$$(\varepsilon)_\infty = (\mathfrak{a}, 0, -\mathfrak{a})\left((-\mathfrak{h}_1, -\mathfrak{h}_1, -\mathfrak{h}_1) + (\mathfrak{a} - \mathfrak{h}_2, -\mathfrak{h}_2, -\mathfrak{a} - \mathfrak{h}_2)\right).$$

We distinguish between the two cases $\mathfrak{h}_1 \neq \mathfrak{a} + \mathfrak{h}_2$ and $\mathfrak{h}_1 = \mathfrak{a} + \mathfrak{h}_2$. In the first case we have $(\varepsilon)_\infty = (?, ?, \min(-\mathfrak{h}_1, -\mathfrak{a} - \mathfrak{h}_2) - \mathfrak{a})$, hence $H_K(\varepsilon) \geq \max(\mathfrak{h}_1, \mathfrak{h}_2 + \mathfrak{a}) + \mathfrak{a} \geq \mathfrak{a}$. In the second case we obtain $(\varepsilon)_\infty = (?, -\mathfrak{h}_1, ?)$, hence $H_K(\varepsilon) \geq \mathfrak{h}_1 = \mathfrak{h}_2 + \mathfrak{a} \geq \mathfrak{a}$.

We are left to the case $h_1 = 0$ or $h_2 = 0$ and obtain

$$\begin{aligned}(\varepsilon)_\infty =&(2\mathfrak{a}, 0, -2\mathfrak{a})(-\mathfrak{h}_2, -\mathfrak{h}_2, -\mathfrak{h}_2) = (?, ?, -2\mathfrak{a} - \mathfrak{h}_2) \quad \text{or}\\ (\varepsilon)_\infty =&(\mathfrak{a}, 0, -\mathfrak{a})(-\mathfrak{h}_1, -\mathfrak{h}_1, -\mathfrak{h}_1) = (?, ?, -\mathfrak{a} - \mathfrak{h}_1),\end{aligned}$$

i.e. in both cases $H_K(\varepsilon) \geq \mathfrak{a}$. ∎

Corollary 3. *There does not exist a rational integer k with $|k| > 1$ and a unit $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ such that $\alpha = \varepsilon^k$. Furthermore, there exists a unit $\varepsilon \in \mathbb{C}[T][\alpha]^\times$ such that α and ε form a fundamental system, i.e. $\mathbb{C}[T][\alpha]^\times = \mathbb{C}\langle\alpha, \varepsilon\rangle$.*

Proof: The first part is obvious, since if such k and ε would exist, then $0 < H_K(\varepsilon) < \mathfrak{a}$, a contradiction to Lemma 5.

Suppose now $(\varepsilon_1, \varepsilon_2)$ is a fundamental system such that $k > 0$ is minimal with $\xi\varepsilon_1^k\varepsilon_2^l = \alpha$ and $\xi \in \mathbb{C}^\times$. Now $k = lq+r$ with $0 \leq r < l$ and $(\varepsilon_1, \varepsilon = \varepsilon_1^q\varepsilon_2)$ is again a fundamental system, such that $\xi\varepsilon_1^r\varepsilon^l = \alpha$. Since we assume k is minimal, we have $r = 0$ and from the first part of the Corollary we deduce $l = 1$, hence (α, ε) is a fundamental system. ∎

Now, Proposition 5 follows easily from Lemma 5 or from Corollary 3.

Proof of Proposition 5: Suppose ε is a unit with $(\varepsilon)_\infty = (e_1, e_2, e_3)$. Multiplying with appropriate powers of α_1 and α_2 one obtains a unit ε' with $(\varepsilon') = (e_1', e_2', e_3')$ and $0 \leq e_1' < \mathfrak{a}$ and $-\mathfrak{a} < e_2' \leq 0$. But now

$$\begin{aligned}H_K(\varepsilon') =& -\min\{0, e_1'\} - \min\{0, e_2'\} - \min\{0, -(e_1' + e_2')\}\\ =& -e_2' + \max\{0, e_1' + e_2'\} = \max\{e_1', -e_2'\} < \mathfrak{a}\end{aligned}$$

implies by Lemma 5 that ε' must be a constant. From this Proposition 5 follows immediately. ∎

Corollary 4. *Let* $\varepsilon \in \mathbb{C}[T][\alpha]^\times$, *then* $H_K(\varepsilon) = n\mathfrak{a}$ *for some* $n \in \mathbb{Z}$.

Proof: From Proposition 5 we now may deduce that all components of $(\varepsilon)_\infty$ are divisible by $\mathfrak{a}$, hence the Corollary follows. ■

6. End of the proof

By Proposition 4 every solution $\beta \in \mathbb{C}[T][\alpha]^\times$ to (7) satisfies $H_K(\beta) \leq 4.67\mathfrak{a} - 2$. Furthermore, we know from Proposition 5 that

$$\beta = \zeta\,(\alpha_1)^{a_1}\,(\alpha_2)^{a_2}\,,$$

where $\zeta \in \mathbb{C}^\times$ and $a_1, a_2 \in \mathbb{Z}$. These two propositions together yield $|a_1|, |a_2| \leq 4$ and $|a_1 - a_2| \leq 4$. By computing all possibilities (there are exactly 61 such pairs (a_1, a_2)), we see that only for $(a_1, a_2) \in \mathcal{E}$, where

$$\mathcal{E} = \{(1,1), (0,0), (1,0)\},$$

we get a β which has the form $x - \alpha y$ with $x, y \in \mathbb{C}[T]$, This yields $\beta \in \mathcal{L}' := \{\zeta(-1-\alpha), \zeta, \zeta\alpha \,:\, \zeta \in \mathbb{C}^\times\}$. Consequently, we have

$$\mathcal{N}_{K/\mathbb{C}[T]}(\beta) = \zeta^3 = \xi \quad (\beta \in \mathcal{L}')$$

and therefore finally Theorem 1. ■

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References

[1] M. Eichler. *Introduction to the theory of algebraic numbers and functions*. Academic Press, New York and London, 1966.

[2] C. Fuchs and V. Ziegler. On a family of Thue equations over function fields. *Monatsh. Math.* To appear (Preprint: `http://www.finanz.math.tu-graz.ac.at/~fuchs/thueff4.pdf`).

[3] B. P. Gill. An analogue for algebraic functions of the Thue-Siegel theorem. *Ann. of Math. (2)*, 31(2):207–218, 1930.

[4] C. Heuberger. Parametrized Thue equations — A survey. available at `http://www.opt.math.tu-graz.ac.at/~cheub/publications/thue-survey.pdf`.

[5] C. Heuberger, A. Pethő, and R. F. Tichy. Thomas' family of Thue equations over imaginary quadratic fields. *J. Symbolic Comput.*, 34(5):437–449, 2002.

[6] C. Heuberger, A. Togbé, and V. Ziegler. Automatic solution of families of Thue equations and an example of degree 8. *J. Symbolic Computation*, 38:1145–1163, 2004.

[7] K. Iwasawa. *Algebraic Functions*, volume 118 of *Translations of Mathematical Monographs Vol.* American Mathematical Society, Providence, Rhode Island, 1993.

[8] G. Lettl. Thue equations over algebraic function fields. *Acta Arith.* To appear.

[9] R. C. Mason. On Thue's equation over function fields. *J. London Math. Soc. (2)*, 24(3):414–426, 1981.

[10] R. C. Mason. *Diophantine equations over function fields*, volume 96 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1984.

[11] M. Mignotte. Verification of a conjecture of E. Thomas. *J. Number Theory*, 44:172–177, 1993.

[12] M. Rosen. *Number theory in function fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[13] W. M. Schmidt. Thue's equation over function fields. *J. Austral. Math. Soc. Ser. A*, 25(4):385–422, 1978.

[14] E. Thomas. Complete solutions to a family of cubic Diophantine equations. *J. Number Theory*, 34(2):235–250, 1990.

[15] A. Thue. Berechnung aller Lösungen gewisser Gleichungen von der Form $ax^r - by^r = f$. *Vid.Skrivter I Mat.- Naturv. Klasse*, 4:1–9, 1918.

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