ON A FAMILY OF THUE EQUATIONS OVER FUNCTION FIELDS

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ABSTRACT. Many families of parametrized Thue equations over number fields have been solved recently. In this paper we consider for the first time a family of Thue equations over a polynomial ring. In particular, we calculate all solutions of

$$X(X-Y)(X-(T+\xi)Y)+Y^3=1+\xi T(1-T)$$

over $\mathbb{C}[T]$ for all $\xi\in\mathbb{C}$.

1. Introduction and Main Result

Let $F\in R[X,Y]$ be a binary irreducible form of degree $d\geq 3$ over some ring R. An equation of the form

$$F(X,Y)=m,\quad m\in R$$

is called a Thue equation, due to Thue who proved the finiteness of solutions in the case of $R=\mathbb{Z}$ in his famous paper [17]. Using the ideas of Thue one may prove that a Thue equation over a ring R that is finitely generated over $\mathbb{Z}$ has only finitely many solutions in R (see [10]), e.g. when R is an order of some number field or $R=\mathbb{Z}[T]$, where T is transcendental. Nowadays it is known how to solve such a Thue equation over R algorithmically. Using Baker's method, the above result was made effective by Győry [3]. Let us mention that for $R=\mathbb{Z}$ the most efficient algorithm is due to Bilu and Hanrot [1].

Now, let K be a function field of characteristic 0, S a finite set of places of K and R the ring of S-integers. By proving an analogue of Baker's method of linear form in logarithms (see the ABC-Theorem e.g. [15, Theorem 7.17]) for function fields, Mason [13, 14] could prove effective bounds for the height of the solutions. Furthermore, he described how to determine effectively all solutions of a given Thue equation over R. Since R is not finitely generated over $\mathbb{Z}$ we cannot hope that such a Thue equation has finitely many solutions, e.g. the equation

$$X^3-(T+1)X^2Y+TXY^2=6T^2(T-1)^2$$

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has infinitely many solutions in $\mathbb{Q}[T]$, namely $(x,y) = ((3c_1-2c_2)T, 3c_1T-2c_2)$, where (c_1,c_2) run through the $\mathbb{Q}$-rational points of the elliptic curve $C_1C_2(3C_1-2C_2) = 1$ (see [12]). This was shown by Lettl [12], who recently considered Thue equations over function fields and gave criteria for which such a Thue equation has infinitely many solutions.

In 1990, Thomas [16] went a step further and considered a parametrized family of Thue equations with positive discriminant. In the last decade, several families of parametrized Thue equations $F_\xi(X,Y) = m$ have been investigated up to degree 8 (cf. [8]). Also more general results have been proved (cf. [5, 6]) and furthermore families over imaginary quadratic number fields were considered (cf. [7]) ; a survey containing further references is given in [4]. Typically, such a family of equations has finitely many families of solutions, that means solutions depending e.g. polynomially on the parameter ξ, and finitely many "sporadic" solutions for certain values of ξ. We mention that it was show by Lettl in [11] that a family of Thue equations can certainly have infinitely many sporadic solutions by considering single Thue equations over function fields which have solutions truely lying in the function field.

The aim of this paper is to determine - for the first time - all solutions of a family of Thue equations over the function field $\mathbb{C}(T)$. We consider the equation

$$X(X-Y)(X-(T+\xi)Y) + Y^3 = 1 + \xi T(1-T),$$

which has the trivial solution $(T,1)$ for every value of $\xi \in \mathbb{C}$. The left-hand-side of the equation is a so-called splitting form, i.e. $X^3-(T+\xi+1)X^2Y+(T+\xi)XY^2+Y^3 = X(X-Y)(X-(T+\xi)Y)+Y^3$ (many of the families studied previously in the integer case are of such a shape; we refer to [4]) and the right-hand-side was chosen such that the equation has at least one non-trivial solution for every value of the parameter. We have chosen the simplest equation for which a non-trivial result can be obtained, especially, in order to see which method is needed to prove the following theorem.

Theorem 1. *The Thue equation*

$$X^3-(T+\xi+1)X^2Y+(T+\xi)XY^2+Y^3 = -\xi T^2+\xi T+1 \tag{1}$$

has only finitely many solutions over $\mathbb{C}[T]$ for all $\xi \in \mathbb{C}$. Let L_ξ be the set of solutions $(x,y) \in \mathbb{C}[T] \times \mathbb{C}[T]$ to (1) for fixed $\xi \in \mathbb{C}$, then

$$L_\xi = \{\zeta(T,1) \,:\, \zeta^3 = 1\},$$

if $\xi \neq -1, 0, 1$. Furthermore,

$$\begin{aligned} L_1 &= \{\zeta(T,1), \zeta(T+1,T) \,:\, \zeta^3 = 1\},\\ L_{-1} &= \{\zeta(T,1), \mu(T-i, T-1-i), \nu(T+i, T-1+i) \,:\, \zeta^3 = 1, \mu^3 = -i, \nu^3 = i\},\\ L_0 &= \{\zeta(T,1), \zeta(1,0), \zeta(1,1), \zeta(0,1), \zeta(1,-T-1) \,:\, \zeta^3 = 1\}. \end{aligned}$$

By the work of Lettl [12, Corollary 2] it is immediate that equation (1) has only finitely many solutions over $\mathbb{C}[T]$ for each $\xi \in \mathbb{C}$, since $X^3-(T+\xi+1)X^2Y+(T+

$\xi)XY^2+Y^3$ is irreducible, as is shown in Section 2. So the main work is to determine those finitely many solutions for each $\xi\in\mathbb{C}$.

In order to prove this Theorem we will follow the original ideas of Mason [14, Chapter 2]. In Section 2 we present some well known results and prove some auxiliary lemmas. In Section 3 we compute the places where ramification occurs in the splitting field of equation (1) and furthermore compute the genus of this splitting field. From these data one obtains a bound for the height of solutions (x,y) to (1) from a theorem of Mason (Proposition 2). By some computations in Section 4 we sharpen these bounds. In particular we prove that $\deg(x),\deg(y)\leq 2$. From this bound we compute in Section 6 all solutions to (1) provided $\xi\neq 0$. The special case $\xi=0$ is treated in Section 5.

2. Auxiliary Results

Let us first remind the ABC-Theorem for function fields (see e.g. [15, Theorem 7.17]).

Proposition 1 (ABC-Theorem)**.** *Let K be a function field with characteristic 0 and genus g_K. Let $u,v\in K^\times$ satisfying $u+v=1$ and put $A=(u)_0$, $B=(v)_0$ and $C=(u)_\infty=(v)_\infty$, where $(\cdot)_0$ denotes the zero divisor and $(\cdot)_\infty$ denotes the polar divisor. Then*

$$\deg A=\deg B=\deg C\leq \max\left(0,2g_K-2+\sum_{P\in\mathrm{Supp}(A+B+C)}\deg_K P\right).$$

If the constant field k is algebraically closed and of characteristic 0, Mason [14, chapter 1, Lemma 2] proved the following special case.

Corollary 1. *Let $H(f):=-\sum_{v\in M_K}\min(0,v(f))$ denote the height of an element $f\in K$ and let $\gamma_1,\gamma_2,\gamma_3\in K$ with $\gamma_1+\gamma_2+\gamma_3=0$. Let $\mathcal{V}$ be a finite set of valuations such that for all $v\notin\mathcal{V}$ we have $v(\gamma_1)=v(\gamma_2)=v(\gamma_3)$, then*

$$H(\gamma_1/\gamma_2)\leq\max(0,2g-2+|\mathcal{V}|).$$

Here we denote the set of all valuations in K by M_K. Usually, M_K denotes the set of places of some field K. Since in the function field case valuations and places are one-to-one we use this notation.

We can deduce Corollary 1 by observing

$$\deg_K P=[O_P/P:k]=[O_P/P:O_p/p][O_p/p:k]=f(P|p)\cdot 1=1,$$

where O_P is a discrete valuation ring with quotient field K, maximal ideal P and $p=P\cap k[T]$. From Puiseux's Theorem (see [2] or [9]) we deduce $f(P|p)=1$. The relation $[O_p/p:k]=1$ is obvious since p is generated by $T-a$ or T^{-1} with $a\in k$.

If $F(X,Y)=m$ is a Thue equation over the ring of integers of some function field L, then Mason [13] could prove an effective bound for the height of solutions (x,y) to (1). By integers we mean the set of all elements in L that may only have negative valuations above ∞. Let us remind Mason's bound on the height of the solutions.

Proposition 2. *Let*

$$F(X,Y) := (X-\alpha_1 Y)\cdots(X-\alpha_d Y) = m$$

be a Thue equation over the ring of integers of some function field L with algebraic closed constant field and characteristic 0. Then all solutions (x,y) satisfy

$$H(x,y) \leq 8H + 2g_L + r - 1,$$

where g_L is the genus of L, r is the number of infinite valuations and H denotes the height of the polynomial $(X-\alpha_1)\cdots(X-\alpha_d)/m$.

For the rest of the paper we define

$$\begin{aligned} F_\xi(X,Y) &:= X^3 - X^2Y(T+\xi+1) + XY^2(T+\xi) + Y^3, \\ f_\xi(X) &:= F_\xi(X,1) = X^3 - X^2(T+\xi+1) + X(T+\xi) + 1. \end{aligned}$$

The polynomial f_ξ is irreducible over $\mathbb{C}(T)$. Otherwise f_ξ must have a root, which is a constant α, because the constant term is 1. Furthermore, we have $f_\xi(\alpha) = 0$ and in particular the coefficient of T in $f_\xi(\alpha)$ must vanish, hence, $\alpha = 0, 1$, but in both cases α is not a root of f_ξ. First we want to compute the Galois group and the splitting field of f_ξ.

Proposition 3. *The Galois group of f_ξ is the symmetric group S_3. Let α_1, α_2, and α_3 be the roots of f_ξ in some algebraic closure of $\mathbb{C}(T)$. Then $L = \mathbb{C}(T)(\alpha_1 - \alpha_2) = \mathbb{C}(T)(\alpha_1, \alpha_2, \alpha_3)$ is the splitting field of f_ξ and the polynomial*

$$\begin{aligned} G_\xi(X) :=& X^6 - \left(2(T+\xi+1)^2 - 6(T+\xi)\right) X^4 + (T^2 - T + \xi(2T-1) + \xi^2 + 1)^2 X^2 \\ & - (T+\xi)^2(T+\xi+1)^2 - 4(T+\xi+1)^3 + 4(T+\xi)^3 + 18(T+\xi)(T+\xi+1) + 27 \end{aligned}$$

is the minimal polynomial of $\alpha_1 - \alpha_2$.

Proof: Let us compute the discriminant Δ_ξ of f_ξ. We get

$$\Delta_\xi^2 = (T+\xi)^2(T+\xi+1)^2 + 4(T+\xi+1)^3 - 4(T+\xi)^3 - 18(T+\xi)(T+\xi+1) - 27.$$

A well known criteria for cubic polynomials says that the Galois group is not the symmetric group, if and only if the square of the discriminant is a square in the ground field. Hence, if the Galois group of f_ξ is not S_3, we would have $\Delta_\xi^2 = (aT^2 + bT + c)^2$ for some $a, b, c \in \mathbb{C}$. Comparing the coefficients of T yields a system of equations in a, b, c and ξ. A short computation shows that this system has no solutions, hence the Galois group is the symmetric group S_3.

Let $L := \mathbb{C}(T)(\alpha_1, \alpha_2, \alpha_3)$ be the splitting field. Obviously, $L \supset M = \mathbb{C}(T)(\alpha_1 - \alpha_2)$. We have

$$\#\left\{\sigma(\alpha_1 - \alpha_2) \,:\, \sigma \in G(L|\mathbb{C}(T))\right\} = 6,$$

otherwise $\alpha_i = \alpha_j$ for $i \neq j$ or $\alpha_i = 0$ for some $i = 1, 2, 3$. Hence, $[M : \mathbb{C}(T)] = 6$ and furthermore $M = L$. Since we know all conjugates of $\alpha_1 - \alpha_2$, it is not difficult to compute the minimal polynomial G_ξ. ∎

3. Ramification

The aim of this section is to compute which places are ramified in $L/\mathbb{C}(T)$ and furthermore to compute the genus g_L of L, where $L = \mathbb{C}(T)(\alpha_1, \alpha_2, \alpha_3)$ is the splitting field of f_ξ. From this we obtain a first upper bound for the degree of solutions (x, y) to (1). In a first step we determine which places are ramified in $K := \mathbb{C}(T)(\alpha_1)$. Note that since $\mathbb{C}$ is algebraically closed all discrete valuation rings with quotient field $\mathbb{C}(T)$ are isomorphic to $O_a := \{f(T)/g(T) : f(T), g(T) \in \mathbb{C}[T], g(a) \neq 0\}$ for some $a \in \mathbb{C}$ or $O_\infty := \{f(T)/g(T) : f(T), g(T) \in \mathbb{C}[T], \deg f \leq \deg g\}$. Valuations corresponding to O_a are denoted by v_a (finite valuations) and the valuation corresponding to O_∞ is denoted by v_∞ (infinite valuation).

Lemma 1. *Let $K = \mathbb{C}(T)(\alpha_1)$. The only places that are ramified in K correspond to valuations v_a with $a \in \mathcal{R} \subset \mathbb{C} \cup \{\infty\}$ and*

$$\mathcal{R} = \left\{ \frac{1}{2} \left(-1 + i^n \sqrt{(-1)^n 13 + 16\sqrt{2}} - 2\xi \right) \ : \ 1 \leq n \leq 4 \right\}.$$

Proof: Assume K is not ramified at valuations that lie over v_a and $a \neq \infty$. By Puiseux's Theorem [2, 9] there exists a formal Power series

$$\alpha(T) := \sum_{n=0}^{\infty} a_n (T-a)^n$$

such that $f_\xi(\alpha(T)) = 0$. On the other hand we know given an equation $f(X,T) = 0$ with f holomorphic and $\frac{\partial f}{\partial x}\big|_{T=a} \neq 0$, then there exists a holomorphic function $X(T) = \sum_{n=0}^{\infty} a_n (T-a)^n$ in an open neighborhood $U \subset \mathbb{C}$ of a, such that $f(X(T), T) = 0$. We conclude that ramification over v_a may only occur if

$$\left.\frac{\partial f_\xi}{\partial X}\right|_{T=a} = 3X^2 - 2X(a+\xi+1) + (a+\xi) = 0 \ \text{ and}$$
$$f_\xi(X)|_{T=a} = X^3 - X^2(a+\xi+1) + X(a+\xi) + 1 = 0.$$

Eliminating X from these equations and solving for a yields $a \in \mathcal{R}$. Computing the Puiseux expansion of f_ξ at some $a \in \mathcal{R}$ we see that one valuation is ramified with index 2 and one valuation is unramified.

We are left to prove that v_∞ is unramified in K. Since $\frac{1}{T} f_\xi(X)$ is "holomorphic at $T = \infty$" we may substitute T by $1/T$ and obtain a new function

$$\tilde{f}_\xi(X) = TX^3 - X^2(T(\xi+1)+1) + X(\xi T + 1) + T$$

that is holomorphic in X and T. From basic calculus we know that there exists a Laurent series $\alpha(T) = \sum_{n=-1}^{\infty} a_n T^{-n}$ with $f_\xi(\alpha(T)) = 0$ if $\left.\frac{\partial \tilde{f}_\xi}{\partial X}\right|_{T=0} \neq 0$. Hence we have to prove the impossibility of

$$\left.\frac{\partial \tilde{f}_\xi}{\partial X}\right|_{T=0} = -2X+1 = 0 \text{ and simultaneously}$$
$$\left.\tilde{f}_\xi(X)\right|_{T=0} = -X^2+X = 0.$$

Hence the valuation v_∞ is not ramified. ■

Now we consider the function field L. Recall that we defined $L = \mathbb{C}(T)(\alpha_1, \alpha_2, \alpha_3)$ to be the splitting field of $f_\xi(X)$.

Proposition 4. *The only places that are ramified in L correspond to valuations v_a with $a \in \mathcal{R} \subset \mathbb{C} \cup \{\infty\}$ and*

$$\mathcal{R} = \left\{ \frac{1}{2}\left(-1 + i^n\sqrt{(-1)^n 13 + 16\sqrt{2}} - 2\xi\right) \ : \ 1 \leq n \leq 4 \right\}.$$

There are exactly three valuations lying above v_a with ramification index 2 for each $a \in \mathcal{R}$. Furthermore $g_L = 1$, where g_L denotes the genus of the function field L.

Proof: Since L/K is a Galois extension the statement concerning the ramification index is obvious from Lemma 1. With similar arguments as in the proof of Lemma 1 we see that there does not occur any further ramification. In order to compute the genus we recall the Hurwitz-Formula [15, Theorem 7.16, page 90]. Let L/K be a finite, geometric extension of function fields of characteristic 0 and let g_K and g_L be the genus of K and L, respectively, then

$$2g_L - 2 = [L:K](2g_K - 2) + \sum_{w \in M_L} e_w, \tag{2}$$

where M_L is the set of valuations of L and e_w denotes the ramification index of w in the extension L/K. If we put $K = \mathbb{C}(T)$, then $g_L = 1$ is computed from (2). ■

We are now able to prove a first bound for the degrees of solutions X and Y to (1).

Corollary 2. *Let (x, y) be a solution to* (1) *with $\xi \neq 0$, then* $\max(\deg x, \deg y) \leq 17$ *and if $\xi = 0$ we have* $\max(\deg x, \deg y) \leq 9$.

Proof: We have to make our computations in L, since in this function field the Thue equation (1) splits and we want to use Proposition 2. From Proposition 4 we know $r = 6$ and $g_L = 1$. Suppose first $\xi \neq 0$. One computes

$$H = \max\left(H\left(\frac{1}{-\xi T^2 + \xi T - 1}\right), H\left(\frac{T+\xi+1}{-\xi T^2 + \xi T - 1}\right), H\left(\frac{T+\xi}{-\xi T^2 + \xi T - 1}\right)\right) = 12.$$

Proposition 2 yields now $\max(H(x), H(y)) \leq 103$. Notice that we assume $x, y \in \mathbb{C}[T]$, hence $\deg x = 6H(x)$ and $\deg y = 6H(y)$, so we have proved the corollary in the case

$\xi \neq 0$. If $\xi = 0$ we similarly compute $H = 6$, hence $\max(H(x), H(y)) \leq 55$, respectively, $\max(\deg x, \deg y) \leq 9$. ■

4. Reduction of Height

From Corollary 2 one could try to compare coefficients and so deduce all solutions. However, the number of equations and unknowns is too large to do this efficiently and therefore, we first calculate a better lower bound for $\deg x$ and $\deg y$. In order to achieve this we use the ABC-Theorem (Proposition 1) and the method of the proof of Mason for the finiteness of a single Thue equation over function fields instead of applying Mason's theorem (Proposition 2) at once. First we will fix some notations usually used in the number field case.

Let us consider the Thue equation

$$F(X, Y) = m \tag{3}$$

over the ring of integers $\mathfrak{o}$ of some function field K. Let L be the splitting field of $F(X, 1)$, thus we have

$$F(X, Y) = (X - \alpha_1 Y) \cdots (X - \alpha_d Y).$$

Let $(x, y) \in \mathfrak{o}^2$ be some solution to equation (3). We put for pairwise distinct indices $i, j, l \in \{1, \ldots, d\}$

$$\begin{aligned} \beta_i(x, y) &= x - \alpha_i y, \\ \gamma_{i,j,l}(x, y) &= \beta_i(x, y)(\alpha_j - \alpha_l) = (x - \alpha_i y)(\alpha_j - \alpha_l). \end{aligned}$$

It is easy to check that

$$\gamma_{i,j,l}(x, y) + \gamma_{j,l,i}(x, y) + \gamma_{l,i,j}(x, y) = 0. \tag{4}$$

This identity is usually called Siegel's identity. Furthermore, the β's are S-integers in the function field L. Moreover, we have from equation (3)

$$\beta_1 \cdots \beta_d = m. \tag{5}$$

4.1. **Preliminaries.** Before we go on we pause for a moment to introduce another notation. Let $M/\mathbb{C}(T)$ be a Galois extension of degree d, $\alpha \in M$ and for each $a \in \mathbb{C} \cup \{\infty\}$ let us fix a d-tuple $(w_1, \ldots, w_d)$ of valuations in M with $w_i | v_a$ for $i \in \{1, \ldots, d\}$, where the valuations are used with multiplicity, i.e. a valuation with ramification index e is written down e times. Let us define

$$(\cdot)_a : M \to \mathbb{Z}^d, \qquad f \mapsto (f)_a := (w_1(f), \ldots, w_d(f)).$$

For every $\sigma \in G(M|\mathbb{C}(T))$ there exists obviously a permutation $\tau(\sigma) \in S_d$ such that

$$(\sigma\alpha)_a = (w_{\tau(1)}(\alpha), \ldots, w_{\tau(d)}(\alpha))$$

and furthermore $T = \{\tau(\sigma) : \sigma \in G(M|\mathbb{C}(T))\}$ is a transitive subgroup of S_d. Let $w | v_a$ have ramification index e and let $f \in \mathbb{C}(T)$ with $v_a(f) = m_a$, then we have $(f)_a = (em_a, \ldots, em_a)$. Collecting these facts together we obtain:

Lemma 2. *Let $\alpha \in M$ be an integer in M and suppose there exists a conjugate $\beta = \sigma\alpha$, with some $\sigma \in G(M/\mathbb{C}(T)) \setminus \{\mathrm{id}\}$ such that there is a valuation $w \nmid v_\infty$ with $w(\alpha) = w(\beta) \neq 0$ and suppose that for the corresponding permutation τ we have $w_{\tau(1)} \neq w_1$. Then the norm $\mathrm{N}(\alpha) = \prod_{i=1}^d \sigma_i\alpha$ has a non constant quadratic factor.*

4.2. **Calculation of the heights.** Let us return to our problem. We assume from now on $\xi \neq 0$. In a first step we want to get a bound for $H\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right)$. Since

$$\beta_1\beta_2\beta_3 = -\xi T^2 + \xi T + 1 = -\xi\left(T - \frac{\sqrt{\xi}+\sqrt{4+\xi}}{2\sqrt{\xi}}\right)\left(T - \frac{\sqrt{\xi}-\sqrt{4+\xi}}{2\sqrt{\xi}}\right)$$

the β's may only have non zero valuations at w if $w|v_a$ or $w|v_\infty$ with

$$a \in \mathcal{B} := \left\{\frac{\sqrt{\xi}+\sqrt{4+\xi}}{2\sqrt{\xi}}, \frac{\sqrt{\xi}-\sqrt{4+\xi}}{2\sqrt{\xi}}\right\}.$$

Since the $\alpha_{i,j} := \alpha_i - \alpha_j$ with distinct $i, j \in \{1,2,3\}$ are roots of G_ξ the $\alpha_{i,j}$ may only have non zero valuations at w if $w|v_a$ or $w|v_\infty$, where a is a complex root of the constant term of G_ξ, i.e. $a \in \mathcal{R}$. We conclude that the $\gamma's$ may only have non zero valuations at w lying above v_a with $a \in \mathcal{R}\cup\mathcal{B}\cup\{\infty\}$. By the considerations above and Proposition 4 there are exactly 30 such valuations. Let us consider Siegel's identity

$$\gamma_{1,2,3} + \gamma_{2,3,1} + \gamma_{3,2,1} = 0$$

and we obtain by Corollary 1 that $H\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) \leq 30$.

Next, we want to compute an upper bound for $H(\beta_1/\beta_2)$. Therefore, we conclude from Lemma 2, that there is no finite valuation w, such that $w(\alpha_{2,3}) = w(\alpha_{3,1}) \neq 0$, otherwise the constant term of G_ξ would have a non constant quadratic factor. From the above considerations we further deduce that $H(\alpha_{2,3}) = H(\alpha_{3,1}) = \frac{1}{6}H_G = 4$, where H_G denotes the height of the constant term of G_ξ. Furthermore, we know that the β's may only have positive finite valuations $w|v_a$ if $a \in \mathcal{B}$. Let us denote by

$$H_a(\alpha) := -\sum_{w|v_a} \min(0, w(\alpha)), \quad a \in \mathbb{C}\cup\{\infty\}$$

the local height. Obviously, we have

$$H(\alpha) = \sum_{a\in\mathbb{C}\cup\{\infty\}} H_a(\alpha). \tag{6}$$

From equation (6) one computes

$$\begin{aligned} H\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right) =&\sum_{a\in\mathcal{R}} H_a\left(\frac{\alpha_{2,3}}{\alpha_{3,1}}\right)+\sum_{b\in\mathcal{B}} H_b\left(\frac{\beta_1}{\beta_2}\right)+H_\infty\left(\frac{\gamma_{1,2,3}}{\gamma_{2,3,1}}\right)\\ =&4+\sum_{b\in\mathcal{B}} H_b\left(\frac{\beta_1}{\beta_2}\right)+H_\infty\left(\frac{\beta_1}{\beta_2}\cdot\frac{\alpha_{2,3}}{\alpha_{3,1}}\right)\\ \geq&4+\sum_{b\in\mathcal{B}} H_b\left(\frac{\beta_1}{\beta_2}\right)+H_\infty\left(\frac{\beta_1}{\beta_2}\right)-H_\infty(\alpha_{3,1})\\ =&H\left(\frac{\beta_1}{\beta_2}\right). \end{aligned} \tag{7}$$

The second equation in (7) holds, since there is no finite valuation w such that $w(\alpha_{2,3})=w(\alpha_{3,1})\neq 0$ as remarked above. Inequality (7) yields now $H(\beta_1/\beta_2)\leq 30$.

Let us consider the valuation structure of β_1. Since $\beta_1\in K$ we have (after a suitable permutation of the 6-tuple of valuations defining $(\cdot)_a$)

$$\begin{aligned} &(\beta_1)_b=(1,1,0,0,0,0) && (\beta_1)_\infty=(-b_1,-b_1,-b_2,-b_2,-b_3,-b_3)\\ &(\beta_2)_b=(0,0,1,1,0,0) && (\beta_2)_\infty=(-b_3,-b_3,-b_1,-b_1,-b_2,-b_2)\\ &(\beta_3)_b=(0,0,0,0,1,1) && (\beta_3)_\infty=(-b_2,-b_2,-b_3,-b_3,-b_1,-b_1)\\ &(\beta_1/\beta_2)_b=(1,1,-1-1,0,0) && (\beta_1/\beta_2)_\infty=(b_3-b_1,b_3-b_1,b_1-b_2,b_1-b_2,b_2-b_3,b_2-b_3) \end{aligned}$$

where $b\in\mathcal{B}$ and $b_1\geq b_2\geq b_3$. Since the sum over all valuations must be zero we have $b_3\leq 0$, $b_1\geq 1$ and $b_1+b_2+b_3=1$. From $H(\beta_1/\beta_2)=2+2(b_1-b_3)\leq 30$ we deduce $14\geq b_1-b_3$. This yields

$$H(\beta_1)=2\cdot\max(b_1,b_1+b_2)\leq 2\cdot\max(14+b_3,14-b_1+1)\leq 28. \tag{8}$$

Next, we want to prove that $H(\alpha_1)=2$. Since the constant term of f_ξ is 1, it is clear that α_1 is a unit in K, respectively in L, hence α_1 has only non zero valuations at $w|v_\infty$. We may assume

$$(\alpha_1)_\infty=(a_1,a_1,a_2,a_2,a_3,a_3),$$

with $a_1\geq a_2\geq a_3$ and since $H(\alpha_1)=H(\alpha_1^{-1})$ we may further assume $a_1,a_2\geq 0$. This yields

$$6=H(T+\xi+1)=H(\alpha_1+\alpha_2+\alpha_3)=6\cdot\min(a_1,a_2,a_3)=3H(\alpha_1),$$

hence $H(\alpha_1)=2$. Another unit is given by α_1-1, since $\alpha_1(\alpha_1-1)(\alpha_1-(T+\xi))=1$. Because of $\alpha_1,\alpha_1-1\in K$, hence $2|H(\alpha_1-1)$, $H(\alpha_1)+H(1)\geq H(\alpha_1-1)$ and α_1-1 is not a constant, we conclude $H(\alpha_1-1)=2$. This yields $H_K(\alpha_1)=H_K(\alpha_1-1)=1$, where H_K denotes the height associated to K. Since $\alpha_1\neq c(\alpha_1-1)$ for any $c\in\mathbb{C}$, we have proved that α_1 and α_1-1 generate the unit group of K factored by $\mathbb{C}^\times$.

Let $\tilde{\beta} \in K$ such that $H(\tilde{\beta}) = 4$ and after a suitable permutation of valuations

$$\begin{aligned}(\tilde{\beta})_b &= (1,1,0,0,0,0), \quad (b \in \mathcal{B}),\\ (\tilde{\beta})_\infty &= (0,0,-2,-2,0,0),\\ (\alpha_1)_\infty &= (1,1,-1,-1,0,0).\end{aligned}$$

Then all β_1's that might yield solutions to (1) are of the form $\tilde{\beta}\alpha_1^{a_1}(\alpha_1-1)^{a_2}$ with $|a_1+a_2|, |a_1|, |a_2| \leq 16$. We want to construct $\tilde{\beta}$. Therefore let us set $\tilde{\beta}_1 := \tilde{\beta}, \tilde{\beta}_2$ and $\tilde{\beta}_3$ the conjugates of $\tilde{\beta}$. We have

$$\tilde{\beta}_i = h_0 + h_1\alpha_i + h_2\alpha_i^2 \quad (1 \leq i \leq 3), \tag{9}$$

with $h_0, h_1, h_2 \in \mathbb{C}(T)$. Solving this linear system by Cramer's rule one obtains

$$\begin{aligned}h_0\delta_\xi^2 &= \delta_\xi\left(\tilde{\beta}_1\alpha_2\alpha_3(\alpha_3-\alpha_2) + \tilde{\beta}_2\alpha_3\alpha_1(\alpha_1-\alpha_3) + \tilde{\beta}_3\alpha_1\alpha_2(\alpha_2-\alpha_1)\right),\\ h_1\delta_\xi^2 &= \delta_\xi\left(\tilde{\beta}_1(\alpha_2+\alpha_3)(\alpha_2-\alpha_3) + \tilde{\beta}_2(\alpha_3+\alpha_1)(\alpha_3-\alpha_1) + \tilde{\beta}_3(\alpha_1+\alpha_2)(\alpha_1-\alpha_2)\right),\\ h_2\delta_\xi^2 &= \delta_\xi\left(\tilde{\beta}_1(\alpha_3-\alpha_2) + \tilde{\beta}_2(\alpha_1-\alpha_3) + \tilde{\beta}_3(\alpha_2-\alpha_1)\right),\end{aligned} \tag{10}$$

where

$$\delta_\xi = \det(\alpha_j^{i-1})_{1\leq i,j\leq 3} = (\alpha_1-\alpha_2)(\alpha_2-\alpha_3)(\alpha_3-\alpha_1)$$

is the discriminant of f_ξ. On the right side of (10) only integers in L occur, hence $h_i\delta_\xi^2 \in \mathbb{C}[T]$ with $i = 1,2,3$. Furthermore, some analysis on the infinite valuations yields $\deg h_i\delta_\xi^2 \leq 4$. This is done by computing each infinite valuation on the right side of (10) using the facts

$$\begin{aligned}&v(\alpha\beta) = v(\alpha) + v(\beta), &&\\ &v(\alpha+\beta) = v(\alpha), &&\text{if } v(\alpha) < v(\beta),\\ &v(\alpha+\beta) \geq v(\alpha), &&\text{if } v(\alpha) = v(\beta).\end{aligned}$$

Further analysis yields

$$\begin{aligned}h_2\delta_\xi^2\alpha_1^{a_1}(\alpha_1-1)^{a_2} = &\delta_\xi\left(x((\alpha_3-\alpha_2)+(\alpha_1-\alpha_3)+(\alpha_2-\alpha_1))\right.\\ &\left.-y(\alpha_1(\alpha_3-\alpha_2)+\alpha_2(\alpha_1-\alpha_3)+\alpha_3(\alpha_2-\alpha_1))\right) = 0,\end{aligned}$$

hence $h_2 = 0$. Now, we have

$$\delta_\xi^2 x - \delta_\xi^2\alpha_1 y = \delta_\xi^2\beta_1 = \left(\sum_{i=0}^{4}\sum_{j=0}^{1} c_{i,j}T^i\alpha_1^j\right)\alpha_1^{a_1}(\alpha_1-1)^{a_2} = H_0 + H_1\alpha_1 + H_2\alpha_1^2, \tag{11}$$

where $c_{i,j} \in \mathbb{C}$ are not specified yet and $H_i \in \mathbb{C}[T]$, $i = 1,2,3$. Obviously, $H_2 = 0$ and $\delta_\xi^2 | H_0, H_1$. By comparing coefficients we obtain for each admissible pair of exponents

(a_1, a_2) a linear system with unknowns $c_{i,j}$. By solving each system we obtain that only if $(a_1, a_2) \in \mathcal{E}$ with

$$\begin{aligned}\mathcal{E} =& \{(-2,1),(-1,-2),(-1,-1),(-1,0),(-1,1),(-1,2),\\ &(0,-1),(0,0),(0,1),(1,-1),(1,0),(2,-1),(3,-2),(3,-1)\},\end{aligned}$$

then the corresponding system has a non-trivial solution. Since a trivial solution yields $\beta_1 = 0$, which is a contradiction, the last result may be reformulated in the following way:

Proposition 5. *We have $H(\beta_1) \leq 10$, i.e. $H_K(\beta_1) \leq 5$.*

From this bound we find now a bound for the degree of solutions (x, y) to (1).

Corollary 3. *Let (x, y) be a solution to* (1) *then* $\max\{\deg(x), \deg(y)\} \leq 2$.

Proof: Since $x, y \in \mathbb{C}[T]$ we have

$$\begin{aligned}(x)_\infty &= (\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x}),\\ (y)_\infty &= (\mathfrak{y},\mathfrak{y},\mathfrak{y},\mathfrak{y},\mathfrak{y},\mathfrak{y}),\\ (y\alpha_1)_\infty &= (\mathfrak{y}+1,\mathfrak{y}+1,\mathfrak{y}-1,\mathfrak{y}-1,\mathfrak{y},\mathfrak{y}),\end{aligned}$$

with $\mathfrak{x} = -\deg x, \mathfrak{y} = -\deg y \leq 0$. In the computation of $(\beta_1)_\infty$ we distinguish 5 cases.

$$(\beta_1)_\infty \geq \begin{cases}(\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x},\mathfrak{x}) & \text{if } \mathfrak{x} < \mathfrak{y}-1,\\ (\mathfrak{x},\mathfrak{x},\infty,\infty,\mathfrak{x},\mathfrak{x}) & \text{if } \mathfrak{x} = \mathfrak{y}-1,\\ (\mathfrak{x},\mathfrak{x},\mathfrak{y}-1,\mathfrak{y}-1,\infty,\infty) & \text{if } \mathfrak{x} = \mathfrak{y},\\ (\infty,\infty,\mathfrak{y}-1,\mathfrak{y}-1,\mathfrak{y},\mathfrak{y}) & \text{if } \mathfrak{x} = \mathfrak{y}+1,\\ (\mathfrak{y}+1,\mathfrak{y}+1,\mathfrak{y}-1,\mathfrak{y}-1,\mathfrak{y},\mathfrak{y}) & \text{if } \mathfrak{x} > \mathfrak{y}+1,\end{cases}$$

where $\geq$ is considered componentwise. From the fact that β_1 is an integer and hence $10 \geq H(\beta_1) = H_\infty(\beta_1)$ and taking into account that $\mathfrak{x}, \mathfrak{y} \in \mathbb{Z}$, we get the following bounds;

$$\begin{array}{lll}\mathfrak{x} \geq -1, & \mathfrak{y} \geq 1, & \text{if } \mathfrak{x} < \mathfrak{y}-1,\\ \mathfrak{x} \geq -2, & \mathfrak{y} \geq -1, & \text{if } \mathfrak{x} = \mathfrak{y}-1,\\ \mathfrak{x} \geq -2, & \mathfrak{y} \geq -2, & \text{if } \mathfrak{x} = \mathfrak{y},\\ \mathfrak{x} \geq -1, & \mathfrak{y} \geq -2, & \text{if } \mathfrak{x} = \mathfrak{y}+1,\\ \mathfrak{x} \geq 1, & \mathfrak{y} \geq -1, & \text{if } \mathfrak{x} > \mathfrak{y}+1.\end{array}$$

■

5. The special case $\xi = 0$

By similar considerations as in Section 4 we want to solve the case $\xi = 0$. Using Mason's version of the ABC-Theorem (Corollary 1) we obtain $H(\gamma_1/\gamma_2) \leq 18$, since the β's are units and have non-zero valuations only above v_∞. The same computations as in Section 4 yield $H(\beta_1) \leq 16$. We further obtain

$$x + \alpha_1 y = \beta_1 = \alpha_1^{a_1}(\alpha_1 - 1)^{a_2} = H_0 + \alpha_1 H_1 + \alpha_1^2 H_2,$$

with $|a_1+a_2|, |a_1|, |a_2| \leq 8$. Obviously, $H_2 = 0$, but this only holds for exponents $(a_1, a_2) \in \mathcal{E}'$ with

$$\mathcal{E}' = \{(-1,-1), (0,0), (0,1), (1,0), (3,-1)\}.$$

The exponents $(a_1, a_2) \in \mathcal{E}'$ determine β_1 up to a constant factor, which is easily computed in each case, hence we have found all solutions in the case of $\xi = 0$.

6. Proof of the main Theorem

Let (x, y) be a solution to (1). By Corollary 3 we may assume

$$x = x_0 + x_1 T + x_2 T^2,$$
$$y = y_0 + y_1 T + y_2 T^2.$$

Substituting this in (1) yields by comparing the coefficients of T^7 the equation $x_2 y_2 (x_2 - y_2) = 0$, hence $x_2 = 0, y_2 = 0$ or $x_2 = y_2$. Let us assume $\max(\deg x, \deg y) = 2$ then we have $y_2 \neq 0, x_2 \neq 0$ and $x_2 = y_2 \neq 0$, respectively. Comparing the coefficients of lower powers of T we get the following table:

T^7	T^6	T^5	T^4	T^3
$x_2 = 0$	$x_1 = y_2$	$x_0 = y_2(1+\xi) - y_1$	$y_0 = y_1(\xi+1) - (\xi+1)^2 y_2$	$y_2 = 0$
$y_2 = 0$	$x_2 = y_1$	$x_1 = x_2 + y_0$	$x_0 = \xi y_0$	$x_2 = 0$
$y_2 = x_2$	$x_1 = y_2 + y_1$	$x_0 = y_0 + y_1 - \xi x_2$	$y_0 = \xi y_1 - \xi^2 x_2 - 2x_2$	$y_1 = 2\xi x_2 + x_2/2$

The first two cases yield contradictions. Comparing further coefficients in the case $y_2 = x_2$ we obtain $9x_2^3/2 = -\xi$ and $-\xi - 2\xi^2 = \xi$, hence $\xi = 0, -1$. Substituting in the constant term yields $1 = 0, -55/36$, hence in both cases a contradiction.

So we may assume x and y are linear, i.e.

$$x = x_0 + x_1 T, \qquad y = y_0 + y_1 T.$$

Comparing the coefficient of T^4 now yields $x_1 y_1 (x_1 - y_1) = 0$, this is $x_1 = 0, y_1 = 0$ or $x_1 = y_1$. By assuming $\max(\deg x, \deg y) = 1$ we have $y_1 \neq 0, x_1 \neq 0$ and $x_1 = y_1 \neq 0$, respectively. Solving the equations that occur by comparing coefficients of lower powers of T, we obtain following table by successive elimination of unknowns:

T^4	T^3	T^2	T^1	$T^0 = 1$
$x_1 = 0$	$x_0 = -y_1$	$y_0 = \frac{\xi y_1^3 + y_1^3 - \xi}{y_1^2}$	$\xi = \frac{2y_1^3}{1-y_1^3}$	$y_1^3 = -1$
$y_1 = 0$	$x_1 = y_0$	$x_0 = \frac{\xi y_0^3 - \xi}{y_0^2}$	$y_0^3 = 1$	
$x_1 = y_1$	$x_0 = y_0 + y_1$	$y_0 = \frac{\xi y_1^3 - \xi}{y_1^2}$	$\xi = 1 \qquad y_1^3 = 1$	
			$\xi = \frac{-y_1^3 + y_1^3\sqrt{-3+4y_1^3}}{2(y_1^3-1)} \qquad y_1^3 \neq 1$	$y_1^3 = \pm i$
			$\xi = \frac{-y_1^3 - y_1^3\sqrt{-3+4y_1^3}}{2(y_1^3-1)} \qquad y_1^3 \neq 1$	$y_1^3 = 1$

We have either a contradiction or a solution, which is listed in Theorem 1.

So we are left to the case that the solution (x, y) is constant, i.e. $x = x_0$ and $y = y_0$ with $x_0, y_0 \in \mathbb{C}$. Substituting this in $F_\xi(x, y)$ yields

$$F_\xi(x,y) = T(x_0y_0^2 - x_0^2y_0) + x_0^3 - x_0^2y_0 - \xi x_0^2y_0 + \xi x_0y_0^2 + y_0^3 = -\xi T^2 + \xi T + 1,$$

again a contradiction. ∎

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