Alpbach Summer School Program

This is a plan for twelve lectures on [12], a new proof of Faltings's Theorem (formerly Mordell's Conjecture). If time permits, the seminar will also cover applications of the method of [12] to families of hypersurfaces.

Recall the Mordell-Faltings theorem.

Theorem 0.1. *If Y is a curve of genus at least two over a number field K, then $Y(K)$ is finite.*

Remark 0.2. *The same argument can also be applied to the S-unit theorem. We'll do that first, as a toy example.*

Faltings's first paper on the subject [8] proves, in order, the following results. (See also [14] for a survey.)

- The Tate conjecture for abelian varieties over a number field: an abelian variety is determined (up to isogeny) by its Tate module.
- The Shafarevich conjecture for abelian varieties (or curves) over a number field: there are only finitely many abelian varieties of fixed dimension g (resp. curves of genus g), defined over a fixed number field K and having good reduction away from a fixed finite set S of primes of K.
- Mordell's conjecture.

Faltings deduces the Tate conjecture from a study of heights of abelian varieties. That Tate implies Shafarevich is a consequence of Faltings's finiteness theorem for Galois representations: there are only finitely many possible Tate modules for an abelian variety. That Shafarevich implies Mordell is shown by constructing a non-isotrivial family of curves X over the given curve Y as base.

The method of [12] is weaker. In particular, it doesn't prove the Shafarevich or Tate conjecture. Instead, [12] fixes a family of curves $X \to Y$, and shows by p-adic methods that the associated Galois representations vary in the family. Specifically, if ρ_y is the p-adic Galois representation $H^1(X_{y,\overline{K}}, \mathbf{Q}_p)$, then under suitable conditions the map $y \mapsto \rho_y$ is finite-to-one. By Faltings's finiteness theorem for Galois representations, this is enough to prove Mordell.

The method of [12] is not specific to H^1; it can be applied just as well to a family of varieties of arbitrary dimension.

We now outline the proof of Mordell-Faltings in [12]. We want to prove $Y(K)$ is finite. We will construct a smooth proper family $X \to Y$; the fiber over each $y \in Y(K)$ will be a finite disjoint union of curves. We will also choose a prime p at which to do p-adic analysis. The assignment $y \mapsto \rho_y = H^1(X_y, \mathbf{Q}_p)$ gives a map from $Y(K)$ to isomorphism classes of p-adic Galois representations. We know from Faltings that there are only finitely many possibilities for ρ_y up to semisimplicity; we will be done if we can show that the assignment $y \mapsto \rho_y^{ss}$ is finite-to-one.

(Of course Faltings also showed that the ρ_y are all semisimple. But since we're trying to give an independent proof of Mordell's conjecture, we're going to pretend we don't know this. For the applications to hypersurfaces, the possible failure of semisimplicity creates real difficulties.

On a first reading, I recommend that the reader ignore the semisimplicity issue entirely, and focus on the proof that $y \to \rho_y$ is finite-to-one.)

Let v be a place of K above p. To any point $y \in Y(K_v)$ we can associate the *local* Galois representation $\rho_y = H^1(X_y, \mathbf{Q}_p)$. We will be done if we can show:

1. The map $y \mapsto \rho_y$ from K_v-points of Y to local Galois representations is finite-to-one.

2. (Ignore on first reading!) There are only finitely many $y \in Y(K_v)$ such that ρ_y can come from a global Galois representation that is not simple.

To prove (1), we use p-adic Hodge theory to study how the Galois representations vary. Specifically, there is an equivalence of categories relating the local Galois representation $H^1(X_y, \mathbf{Q}_p)$ with the filtered ϕ-module $(V, \phi, \mathrm{Fil}^1 V) = H^1_{\mathrm{cris}}(X_y)$. As y varies p-adically, the pair (V, ϕ) is locally constant, and $\mathrm{Fil}^1 V$ varies. The map $y \mapsto \mathrm{Fil}^1 V$ gives a *period map* Φ from a residue disk in Y to a Grassmannian. We need to know that different y give rise to nonisomorphic filtered ϕ-modules; for this, we need to know that $\Phi(Y)$ is not contained in any orbit of the centralizer of ϕ. We deduce this from the following:

3. The image of the period map is Zariski dense in the appropriate Grassmannian. This follows from a monodromy calculation in classical topology.

4. The centralizer of ϕ is not too big. This follows from the semilinearity of ϕ, and the fact that a certain field extension is of large degree.

To prove (2), we show that a generic local Galois representation cannot come from a global Galois representation that is not simple, and then we apply (3).

Other seminars on this topic:
http://math.mit.edu/nt/index_stage
http://www.mi.fu-berlin.de/en/math/groups/arithmetic_geometry/research_seminar/Mordell_wise1819.html
http://www.esaga.uni-due.de/ws1819/alggeo/

Numbering is as in https://arxiv.org/pdf/1807.02721.pdf

0. Introduction. BL will give this lecture.

   This introductory talk will explain the setup of the proof, as above.

1. Introduction to p-adic Hodge theory, first lecture.

   State the main results of crystalline cohomology and the crystalline comparison isomorphism of p-adic Hodge theory. (It's enough to work on the rational level; we won't need integral structures.) In particular, state the comparison between crystalline cohomology with de Rham cohomology, which makes de Rham cohomology into a filtered ϕ-module. Also explain the compatibility with the Gauss-Manin connection. For the comparison isomorphism, what people will need to know is that $H^n_{\mathrm{dR}}(X)$ and $H^n_{\mathrm{et}}(X)$ determine each other, where X is a variety over an unramified extension of $\mathbf{Q}_p$, having good reduction at p.

   It's probably also a good idea to give some formal development of the theory, though I'm not sure how much it will contribute to the audience's understanding.

   For crystalline cohomology, one should perhaps give a definition of the crystalline site. I recommend [7] for an overview. A good thorough reference is [4]. Details can be found in [3], and a short proof of the crystalline-de Rham comparison in [5], but it's probably not necessary to go into this level of detail.

   For p-adic Hodge theory, one should discuss the role of period rings, as in [10] or [6]. Again, I don't think it's realistic to fully develop the theory.

2. Introduction to p-adic Hodge theory, second lecture.

   (The material I have here might not fill up a full lecture, so there is room for overflow from the first lecture.)

   The goal of this lecture is to show what the crystalline-étale comparison looks like for elliptic curves of (good) ordinary reduction.

   Consider E an elliptic curve over $\mathbf{Q}_p$ of ordinary good reduction. Vary E in a p-adic family, while keeping its special fiber (reduction mod p) constant. The cohomology $H^1(E)$ (in either the crystalline or étale theory) is an extension of two one-dimensional objects. The extension class varies with E, but the isomorphism class of $H^1(E)$ is generically constant. Show this both on the Galois representation side and the filtered ϕ-module side.

   On the other hand, suppose now K is a nontrivial unramified extension of $\mathbf{Q}_p$, and consider elliptic curves E/K having given reduction mod p. Here the isomorphism class of $H^1(E)$ varies, though it is not quite a fine enough invariant to determine E. Show this on the filtered ϕ-module side only.

   If time permits, at the end of the lecture, cover the technical lemmas 3.1 and 3.2, showing that Zariski density of the p-adic period map is equivalent to Zariski density of the complex period map.

3. The S-unit equation (§4). This is included as a warm-up, to showcase the main ideas.

Explain the elementary reduction from the S-unit theorem to Lemma 4.2. Give the (straightforward) construction of the modified Legendre family. Assuming Lemmas 4.3 and 4.2, complete the proof of Lemma 4.2 – this is essentially the "cartoon argument" given in the introductory talk. Finally, determine the monodromy of the modified Legendre family (Lemma 4.2), and give the proof of generic simplicity (Lemma 4.3).

In particular, draw attention to the following three ingredients.

- big monodromy (Lemma 4.3),
- small Frobenius centralizer, which follows from the large field extension (bottom of p. 22), and
- generic simplicity (Lemma 4.4).

The argument for Mordell-Faltings will have the same structure.

4. Construction of the Kodaira-Parshin family. This is Section 7, especially 7.3. It's quite easy to construct this Kodaira-Parshin family complex analytically; the difficulty is to show that it is in fact defined over the number field K.

   We work with the Galois covers because it makes our argument in Section 7.3 easier. For the lectures, it might be preferable to only work with the degree-q covers, construct them complex-analytically, and simply state without proof that the family is defined over K.

   Our Kodaira-Parshin family should be considered a generalization of the Hurwitz schemes in [11], which tells a nice, readable complex-analytic story. On the other extreme, [13, §3.22] contains some very general results, and the result of this section could probably be deduced from the arguments there.

5. Friendly places and generic simplicity.

   The goal here is to explain how we deal with the semisimplicity hypothesis in Faltings's finiteness lemma (Lemma 2.3.) The content is in §2.4-2.5. The goal is Lemma 2.10, which has the following consequence: any global Galois representation, whose restriction to a friendly place v is sufficiently generic, must be simple.

   Give the definition of "friendly places" and prove the key result (Lemma 2.8). Deduce as a consequence Lemmas 2.9 and 2.10, and explain how Lemma 2.10 applies in conjunction with Faltings finiteness (Lemma 2.3).

6. The main argument (§5 - 6), first lecture.

   Explain the precise sense in which the Kodaira-Parshin family gives large field extensions. This is the notion of "size_v" of Definition 5.2, as well as the inequality in displayed equation (5.4).

   The prime q will be chosen so that the q-th roots of unity generate a "sufficiently large" extension of K. The point is that the q-torsion in

Jac Y gives a lower bound for the fields arising as H^0 of the Kodaira-Parshin fibers. Next, the place v (in other words, the prime p) will be chosen so that

- the field extension $K[\zeta_q]/K$ gives a high-degree unramified extension of K_v, and
- v is *friendly*, as discussed in §2.4-2.5.

This happens in the proof of Theorem 5.4.

Using these ideas, give the proof of Mordell-Faltings, assuming Proposition 5.3 and the properties of the Kodaira-Parshin family, listed immediately below Proposition 5.3. (This is essentially Theorem 5.4.)

The proof of Proposition 5.3 will occupy the rest of this lecture and all of the next one.

Finish this lecture with the discussion of indexing in the beginning of the proof of Proposition 5.3.

7. The main argument (§5 - 6), second lecture.

   Give the proofs of Lemmas 6.2 and 6.1, which together complete the proof of Proposition 5.3.

   Start with Lemma 6.2, which shows that Galois representations vary in the family, so only finitely many y can give rise to a single isomorphism class of Galois representation. The main idea is that the Frobenius centralizer is small, by Lemma 2.1 and our choice of a large field extension.

   The rest of the lecture will be occupied with the more technical Lemma 6.1.

8. Monodromy, first lecture. The aim of these two lectures is to prove Theorem 8.1 of [12]: large monodromy of the Kodaira–Parshin family. In these lectures we'll work exclusively in the classical topological category. What was a "curve" is now a "surface."

   Begin with a review of the mapping class group of a surface, and introduce Dehn twists. A good reference is [9].

   The mapping class group acts on the homology $H^1(Y, \mathbf{Z})$, giving a map $MCG(Y) \to (Sp)(H^1(Y, \mathbf{Z}))$. If time permits, it might be good to present the proof in [9] that this map is surjective; the ideas will be important later.

   Next, give the setup of Theorem 8.1, where the curve Y has a number of finite covers Z_i, each ramified above the single point $y \in Y$. Explain the action of a finite-index subgroup of $MCG(Y - \{y\})$ acts on Z_i; this is roughly the content of Section 8.2. Explain also the reduction of Theorem 8.1 to Lemma 8.7 and, if time permits, Lemma 8.9.

9. Monodromy, second lecture. This will be the proof of Theorem 8.1.

There are two important pieces of technical input. The first is a study of the monodromy action of Dehn twists on Y (Section 8.3). The second is a study of the structure of Aff_q-covers, and in particular the "normal form" given in Section 8.4. Once this has been understood, the big monodromy result follows from a "mere calculation" (or perhaps even a "proof by picture"), as in Section 8.6.

If time is tight, the details of Lemmas 8.6, 8.8, 8.11 could perhaps be omitted, with some hand-waving justification of why they are "easy" given our preparations.

10. Hypersurfaces, part one: introduction. In the last three lectures we're going to prove a Shafarevich-type result for hypersurfaces (Theorem 10.1).

    Start with a statement of the problem: $\pi\colon X \to Y$ is a smooth proper morphism over $\mathbf{Z}[S^{-1}]$, and we want to bound $Y(\mathbf{Z}[S^{-1}])$. When $\dim Y > 1$, we need a way to control sets of the form $\Phi^{-1}(Z)$, where Φ is the (p-adic) period map and Z is a closed subset of the Grassmannian.

    State Theorem 9.1 (the transcendence theorem of Bakker and Tsimerman, [2]) and Corollary 9.2. State or prove Lemma 9.3, which relates p-adic to complex transcendence. (Lemma 9.3 is analogous to Lemmas 3.1 and 3.2.)

    State the general Theorem 10.1, which gives conditions under which we can conclude that $Y(\mathbf{Z}[S^{-1}])$ is not Zariski dense in Y. Prove Proposition 10.2, which shows that Theorem 10.1 applies to hypersurfaces within a certain range. This is a matter of estimating the Hodge numbers of hypersurfaces. (See [1, Prop. 17.3.2] for formulas to compute the Hodge numbers.) Here also it must be mentioned that the universal family of hypersurfaces has big monodromy.

11. Hypersurfaces, part two: reduction of Theorem 10.1 to Proposition 10.6.

    This talk has two technical goals. The first is to explain how the Frobenius centralizer is controlled in this context (Lemma 10.4). The trick is to use the Hodge torus to bound from below the image of the *global* Galois representation. The second is to reduce Lemma 10.5 to Proposition 10.6. This entails a careful analysis of what happens when two global Galois representations have the same semisimplification, again using Lemma 2.9 to relate the existence *global* subrepresentations to the structure of the *local* representation.

    At the end of the talk it should be clear that the technical linear algebra Proposition 10.6 is all that is needed to complete the proof.

12. Hypersurfaces, part three: combinatorics on reductive groups, proof of Proposition 10.6. Give the proof of Proposition 10.6, which is the content of Section 11.

    We have a vector space V (which should be thought of as the filtered ϕ-module attached to $H^n(X)$) as well as filtered vector spaces A_i (which should be thought of as coming from the factors of the semisimplification of the global Galois representation). We want to bound the number of filtrations F on V such that (V, F) is a repeated extension of the A_i's.

Let $\mathfrak{f}$ be the purported filtration on V, for which the successive quotients are the A_i's. The dimension count we want to bound will be determined combinatorially from the dimensions of $\mathfrak{f}^i \cap F^j V$. To make this manageable, it's convenient to write it all in terms of reductive groups. This happens in sections 11.1-11.4.

The condition coming from Lemma 2.9 is translated in Section 11.4 into the notion of "balanced filtration" – loosely speaking, this condition means that some of the intersections $\mathfrak{f}^i \cap F^j V$ must have dimension much larger than the "generic" dimension. In terms of root systems, the condition of being "balanced" is expressed by equation 11.7. Equation 11.13 is a special case of equation 11.7, and this one special case is used to complete the proof.

References

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