WORKSHOP ON p-ADIC PERIODS

THE SPEAKERS

Abstract. These are notes from the Workshop on p-adic periods in Alpbach, Austria organized by the ETH and the Universität Zürich and meeting from July 18th – 23rd, 2010. Many of the talks were expositions of or heavily based on the notes of Berger. However, many speakers did synthesize enough information in their talks in the form of motivations and expansions that the notes may still prove useful in their own right.

Contents

Note. Please attribute errors first to the scribe.

1. Ramification filtrations and the Ax-Sen-Tate theorem

1.1. **Notation.** There is a lot of notation which will be convenient to fix at the outset.
We assume:

The workshop organizers were G. Wüstholz (Chair), A. Kresch, C. Fuchs who worked with special help of Laurent Berger within the ProDoc module Arithmetic and Geometry. The speakers were Rafael von Känel, Jonathan Skowera, Claudia Scheimbauer, Lars Kühne, Joseph Ayoub, Daniel Haase, Aleksander Momot, Hiep Pham, Thomas Preu, Jun Yu, Mingxi Wang, Philipp Habegger, Giovanni Di Matteo, Sergey Gorchinskiy, Laurent Berger, Brent Doran, and Sergey Rybakov. The notes were recorded by Jonathan Skowera.

- K is a complete discrete valuation field.
- $L/K'/K$ are finite Galois extensions.
- $\overline{L}/\overline{K'}/\overline{K}$ are separable extensions.

Then a lemma says there are unique valuations on K' and L extending the valuation on K. Another lemma says the valuation rings $\mathcal{O}'_K$ of K' and $\mathcal{O}_L$ of L are the integral closures of $\mathcal{O}_K$ in K' and L respectively. A third lemma says there are $y \in \mathcal{O}'_K$ and $x \in \mathcal{O}_L$ such that $\mathcal{O}'_K = \mathcal{O}_K[y]$ and $\mathcal{O}_L = \mathcal{O}_K[x]$.

We define the higher unit groups of L to be

$$U_L^{(n)} := 1 + \mathfrak{m}_L^n.$$

Of course, the definition would work for K' and K as well.

1.2. **Lower ramification filtration.** The action of the Galois group G descends to quotients $\mathcal{O}_L/\mathfrak{m}_L^n$ (Why? Automorphisms of L fixing K map integral elements over $\mathcal{O}_K$ to integral elements over $\mathcal{O}_K$, and invertible elements ($\mathcal{O}_L \setminus \mathfrak{m}_L$) are mapped exactly to invertible elements.)

Definition 1.1. Define the lower ramification filtration of G using this action by

$$G_n := \{g \in G \mid g \text{ acts trivially on } \mathcal{O}_L/\mathfrak{m}_L^{n+1}\},$$

for $n \in \mathbb{Z}_{\geq -1}$.

Then

$$G = G_{-1} \rhd G_{n_1} \rhd \cdots \rhd G_{n_k} = \{1\}$$

(there should be not equal signs under each normal symbol!) for some $n_i \in \mathbb{N}$ which are commonly called the i-th ramification numbers.

We can come by an equivalent definition of the filtration because

$$\begin{aligned} g \text{ acts trivially on } \mathcal{O}_L/\mathfrak{m}_L^{n+1} &\iff ga - a \in \mathfrak{m}_L^{n+1} \quad \forall a \in \mathcal{O}_L \\ &\iff \nu_L(ga - a) \geq n + 1 \quad \forall a \in \mathcal{O}_L \\ &\iff \nu_L(gx - x) \geq n + 1 \end{aligned}$$

Remember the notation that $\mathcal{O}_L = \mathcal{O}_K[x]$. The last line follows because $\nu_L(ga-a) \geq \nu_L(gx-x)$ for all $a \in \mathcal{O}_L$. So we could equivalent define the filtration by

$$G_n = \{g \in G \mid \nu_L(gx - x) \geq n + 1\}$$

The filtration is compatible with the "top" of subextensions but not the "bottom":

$$H_n = H \cap G_n$$

but, generally speaking,

$$(G/H)_n \neq \pi(G_n) \qquad \pi : G \twoheadrightarrow G/H$$

It will be convenient to write the right hand side as G_nH/H, the set of all G_n cosets of H.

1.3. **Upper ramification filtration.** Now we'll look for a filtration that's compatible with the "bottom" of subextensions. To this end, we'll make two preparations.

First, we'll allow any $u \in \mathbb{R}_{\geq -1}$ to appear in the subscript of the filtration by using the second definition given above. This works since $\nu_L(gx - x) \geq u + 1$ makes sense for non-integral u.

Second, we'll simplify notation by introducing a function

$$\begin{aligned} i_G : G &\to \mathbb{N} \cup \{\infty\} \\ g &\mapsto \nu_L(gx - x) \end{aligned}$$

The definition does not depend on the choice of generator x, and could also be defined as

$$i_G(g) = \sup\{n \mid g \text{ acts trivially on } \mathcal{O}_L/\mathfrak{m}_L^n\}.$$

The second definition is useful in proving properties of i_G below. With this notation, the ramification groups can be written as

$$G_u = \{g \in G \mid i_G(g) \geq u + 1\}.$$

The strategy is now to find the relation of u' to u where

$$G_u H/H = (G/H)_{u'}$$

which will be solved be relating $i_{G/H}$ to i_G.

Theorem 1.2. *For any $\hat{g} \in \sigma \in G/H$,*

$$i_{G/H} = \frac{1}{e_{L/K'}} \sum_{h \in H} i_G(\hat{g}h).$$

Proof (Tate). Case $\sigma = 1$: Both sides are $+\infty$.

Case $\sigma \neq 1$: The proposition is equivalent to

$$e_{L/K'} i_{G/H}(\sigma) = \sum_{h \in H} i_G(\hat{g}h)$$

Now $e_{L/K'} i_{G/H}(\sigma) = \nu_L(\sigma y - y) = \nu_L(\hat{g}y - y)$ and $\sum_{h \in H} i_G(\hat{g}h) = \nu_L(\prod_{h \in H}(\hat{g}hx - x))$. The statement is then equivalent to

$$\langle gy - y \rangle = \left\langle \prod_{h \in H}(ghx - x) \right\rangle = \mathfrak{m}_L^r.$$

This is true, but the proof is omitted. □

We'll want to rewrite the terms in the sum using the relation $i_G(\hat{g}h) = \inf\{i_G(h), i_G(\hat{g})\}$, where we now need to choose a special $\hat{g}$, in particular,

$$\hat{g} := \arg\max_{g \in \sigma} i_G(g).$$

This relation follows from two cases.

Case $i_G(h) \geq i_G(\hat{g})$: Then

$$i_G(\hat{g}) \geq i_G(\hat{g}h) \geq \inf\{i_G(\hat{g}), i_G(h)\} = i_G(\hat{g}),$$

where the first relation is by the definition of $\hat{g}$, and the second relation follows by considering the definition of i_G in terms of action on $\mathcal{O}_L/\mathfrak{m}_L^n$. In this case, $i_G(\hat{g}h) = i_G(\hat{g})$.

Case $i_G(h) < i_G(\hat{g})$: Then we immediately have $i_G(\hat{g}h) = i_G(h)$, again by examing when the actions on the quotient rings are trivial.

We've now found the relation between u and u':

$$\begin{aligned}(G/H)_{u'} &= \{\sigma \in G/H \mid i_{G/H}(\sigma) \geq u'+1\} \\ &= \left\{\sigma \in G/H \mid \frac{1}{e_{L/K'}} \sum_{h \in H} \inf\{i_H(h), i_G(\hat{g})\} \geq u'+1\right\}\end{aligned}$$

Define Herbrand's function for a Galois extension L/K as

$$\phi_{L/K}(u) := \frac{1}{e_{L/K}} \sum_{g \in G} \inf\{i_G(g), u+1)$$

This has a simpler form. First note that $e_{L/K} = |G_0|$ since G_0 is the inertia group. Second, take the derivative of $\phi_{L/K}$ at u such that $n < u < n+1$ for some $n \in \mathbb{Z}_{\geq 1}$. This is

$$\phi'_{L/K}(u) = \frac{\#\text{ of } g \in G \text{ such that } i_G(g) \geq n+2}{|G_0|} = \frac{|G_{n+1}|}{|G_0|} = \frac{1}{[G_0 : G_{n+1}]}$$

This suggests the traditional definition of

$$\phi_{L/K}(u) := \int_0^u \frac{dt}{[G_0 : G_t]},$$

where we interpret $[G_0 : G_t]$ as $1/[G_t : G_0]$ for $t < 0$. This is a piece-wise linear, increasing, concave function with kinks at some integers. It is equal to u on the interval $[-1, 0]$.

With the notation of Herbrand's function, we may now write

$$G_u H/H = (G/H)_{\phi_{L/K'}(u)}.$$

Finally, let $\psi_{L/K'}(u) := \phi_{L/K'}^{-1}(u)$ be the inverse of Herbrand's function.

Definition 1.3. The upper ramification filtration is defined as

$$G^u := G_{\psi_{L/K}(u)}.$$

This indeed satisfies the desired property:

$$\begin{aligned}(G/H)^u &= (G/H)_{\psi_{K'/K}(u)} \\ &= G_{\psi_{L/K'}(\psi_{K'/K}(u))} H/H \\ &= G_{\psi_{L/K}(u)} H/H \\ &= G^u H/H\end{aligned}$$

The third equality comes from the relation $\psi_{L/K} = \psi_{L/K'} \circ \psi_{K'/K}$, or equivalently, from $\phi_{L/K} = \phi_{K'/K} \circ \phi_{L/K'}$. The second relation can be seen by comparing the derivatives of both sides:

$$\begin{aligned}
(\phi_{K'/K} \circ \phi_{L/K'})'(u) &= \phi'_{K'/K}(\phi_{L/K'}(u)) \cdot \phi'_{L/K'}(u) \\
&= \frac{1}{e_{K'/K}} |(G/H)_{\phi_{L/K'}(u)}| \cdot \frac{1}{e_{L/K'}} |H_u| \\
&= \frac{1}{e_{L/K}} |G_u H/H||H_u| \\
&= \frac{|G_u H| \cdot |G_u \cap H|}{|H_0| \cdot |H|} \\
&= \frac{|G_u|}{|G_0|} = \phi'_{L/K}(u)
\end{aligned}$$

1.4. **Properties of the filtrations.**

- $G_0 = G^0$ is the inertia group (of the only prime ideal $\mathfrak{m}_L$).
- The fixed field L^{G_0} is the maximal unramified extension of K in L.
- L/L^{G_0} is a totally ramified (at $\mathfrak{m}_L$) extension.
- $$\begin{aligned} G/G_0 &= G/\{g \in G \mid g \text{ trivial on } \mathcal{O}_L/\mathfrak{m}_L = \overline{L}\} \\ &= \mathrm{Gal}(\overline{L}/\overline{K}) \end{aligned}$$
- $U_L/U_L^1 \cong \overline{L}^*$.
- $n \geq 1$:
$$\begin{aligned} G_i/G_{i+1} \hookrightarrow U_L^n/U_L^{n+1} &\cong \mathfrak{m}_L^n/\mathfrak{m}_L^{n+1} \cong \overline{L} \\ g \mapsto \frac{g\pi_L}{\pi_L} \quad u &\mapsto u - 1 \end{aligned}$$
The first morphism is well-defined because $g \in G_i \iff g\pi_L/\pi_L$:
$$n + 1 \leq i_G(g) = \nu_L(g\pi - \pi) = \nu_L(\pi) + \nu_L\left(\frac{g\pi}{\pi} - 1\right) = 1 + \nu_L\left(\frac{g\pi}{\pi} - 1\right),$$
which is true iff $g\pi/\pi \in \mathfrak{m}_L^n + 1 = U_L^{(n)}$. To see it is a homomorphism, use the decomposition
$$\frac{gh\pi}{\pi} = \frac{g\pi}{\pi} \cdot \frac{h\pi}{\pi} \cdot \frac{g\left(\frac{h\pi}{\pi}\right)}{\left(\frac{h\pi}{\pi}\right)}.$$
The last isomorphism is true because $\mathfrak{m}_L^n/\mathfrak{m}_L^{n+1}$ is a 1-dimensional vector space over $\overline{L}$.
Furthermore, if $\Omega_L := (\mathfrak{m}_L/\mathfrak{m}_L^2)$, then $U_L^{(i)}/U_L^{(i+1)} \cong \Omega_L^{\otimes n}$.
- If char $\overline{L} = p > 0$, then
$$G_0 \cong \mathbb{Z}/m\mathbb{Z} \ltimes \widetilde{G}, \qquad |\widetilde{G}| = p^k,$$
for some $k \in \mathbb{N}$.
If char $\overline{L} = 0$, then
$$G_0 \cong \mathbb{Z}/n\mathbb{Z}, \qquad G_1 = \{1\}$$

for some $n \in \mathbb{Z}$.

1.5. **The completion is algebraically closed.** Define $\mathbb{C}_p := \widehat{\mathbb{Q}_p^{\mathrm{alg}}}$, where the $\widehat{\cdot}$ denotes completion in the p-adic norm. One might wonder if, after taking the completion of the algebraic closure, the field is no longer algebraically closed, but thankfully in this case it is by a theorem.

Theorem 1.4. *Let K be a complete valued field. Then $\widehat{K^{alg}}$ is algebraically closed.*

The proof relies on a reduction to the finding of a root of a monic polynomial in $\mathcal{O}_{\widehat{K^{\mathrm{alg}}}}[x]$, and then applying Hensel's lemma and induction on the degree of the polynomial. See Berger Theorem 3.3.1 for more details.

1.6. **The Ax-Sen-Tate theorem.**

Theorem 1.5 (Ax-Sen-Tate)**.** *If L is a complete p-adic field and Galois extensions $L \subset K \subset L^{alg}$, then completion is compatible with taking the fixed field of the Galois action, i.e.,*

$$\widehat{L^{alg}}^{Gal(L^{alg}/K)} = \widehat{K},$$

where the Galois action has been extended by continuity.

Proof. For simplicity, we'll use the notation $G_K := \mathrm{Gal}(L^{\mathrm{alg}}/K)$.

Case $\widehat{K} \subset \widehat{L^{\mathrm{alg}}}^{G_K}$: True.

Case $\widehat{K} \supset \widehat{L^{\mathrm{alg}}}^{G_K}$: Let $\alpha \in \widehat{L^{\mathrm{alg}}}^{G_K}$, so by completeness, $\alpha = \lim_n \alpha_n$ for some $\alpha_n \in L^{\mathrm{alg}}$. Let

$$\Delta_K : L^{\mathrm{alg}} \to \mathbb{Z} \cup \{\infty\}, \qquad \alpha \mapsto \sup_{g \in G_K} |g\alpha - \alpha|_p.$$

be the maximal distance from α to any other root in its minimal polynomial. Then

$$\lim_{n\to\infty} \Delta_K(\alpha_n) = 0.$$

The key lemma, which we prove below, says that for each α_n there exists a $\beta_n \in K$ such that $|\alpha_n - \beta_n| \le c_p \Delta_K(\alpha_n)$ for a fixed constant c_p. Thus

$$\lim_{n\to\infty} |\alpha_n - \beta_n|_p = 0$$

and

$$\alpha = \lim \beta_n \in \widehat{K}.$$

□

It remains to prove the key lemma.

Lemma 1.6 (Ax)**.** *Again, let L be a complete p-adic field and $L \subset K \subset L^{alg}$. For all $\alpha \in L^{alg}$, there exists an approximating $\beta \in K$ whose distance from α is*

$$|\alpha - \beta|_p < c_p \Delta_K(\alpha)$$

for $c_p := p^{p/(p-1)^2}$.

Proof. First let $c_{p,i} := p^{1/(p^{i+1}-p^i)})$. In fact, we show that there exists a $\beta \in K$ such that

$$|\alpha - \beta|_p \le \left(\prod_{i=1}^{\ell(n)} c_{p,i}\right) \Delta_K(\alpha) \le \left(\prod_{i=1}^{\infty} c_{p,i}\right) \Delta_K(\alpha) = c_p \Delta_K(\alpha),$$

where $n := [K(\alpha) : K]$ is the degree of α and $\ell(n) := \max\{\ell \mid p^\ell \leq n\}$. The proof is by induction on the degree n of α:

Case $n = 1$: It's true.

Case $n > 1$: To apply induction, we'll want to reduce to the case of some β of degree less than α. That means bounding $|\beta-\alpha|_p$ in terms of $\Delta_K(\alpha)$, which is the upper bound of $|\alpha'-\alpha|_p$ for roots α' of the minimal polynomial of α. The idea is to view $\alpha' - \alpha$ as a root of the translated minimal polynomial whose roots are all bounded by $\Delta_K(\alpha)$, and find $\beta - \alpha$ as a root of a derivative of the (translate of) the minimal polynomial. For this strategy to succeed, we'll need a relation between bounds on the zeros of a polynomial and bounds on the zeros of its derivatives. The following lemma does the trick.

Lemma 1.7. *If $f \in \overline{L}[x]$ is monic and $|\gamma|_p \leq \delta$ for all roots γ of f, and write $\deg f = p^k d$ for $2 \leq d \leq p$. Then there exists a $\xi \in K$ such that*

$$f^{(p^k)}(\xi) = 0, \quad \textit{and} \quad |\xi|_p \leq c_{p,k}\delta.$$

Let $p(x) := m_\alpha(x+\alpha)$ where m_α is the minimal polynomial of α. Then by the lemma, there exists a $\gamma \in (L^{\text{alg}})^{G_K} = K$ such that $p^{(p^k)}(\gamma') = 0$ and $|\gamma'|_p \leq c_{p,k}\Delta_K(\alpha)$. Define $\gamma := \gamma' + \alpha$. Then

$$m_\alpha^{(p^k)}(\gamma) = 0 \quad \text{and} \quad |\gamma - \alpha|_p \leq \Delta_K(\alpha).$$

It follows that if $g \in G_K$, then

$$\begin{aligned} |g(\gamma)-\gamma|_p &= |g(\gamma) - g(\alpha) + g(\alpha) - \alpha + \alpha - \gamma|_p \\ &\leq \sup\{|g(\gamma-\alpha)|_p, |g(\alpha)-\alpha|_p, |\gamma-\alpha|_p\} \end{aligned}$$

But $|g(\gamma-\alpha)|_p = |\gamma-\alpha|_p \leq c_{p,k}\Delta_K(\alpha)$. Also, $|g(\alpha)-\alpha|_p \leq \Delta_K(\alpha)$ by definition, so

$$|g(\gamma)-\gamma| \leq c_{p,k}\Delta_K(\alpha)$$

and hence

$$\Delta_K(\gamma) \leq c_{p,k}\Delta_K(\alpha).$$

Then by induction there exists $\beta \in K$ such that $|\beta-\gamma|_p \leq \prod_{i=1}^{k-1} c_{p,i}\Delta_K(\alpha)$, so

$$\begin{aligned} |\beta-\alpha|_p &\leq \left(\prod_{i=1}^{k-1} c_{p,i}\right)\Delta_K(\gamma) \\ &\leq \left(\prod_{i=1}^{k} c_{p,i}\right)\Delta_K(\alpha) < c_p\Delta_K(\alpha). \end{aligned}$$

□

2. Witt Vectors

Claudia on Monday, 19th of July, 2010.

2.1. **Motivation.** We may write every element of $\mathbb{Z}_p$ as an infinite sum with integer coefficients between 0 and $p-1$. In other words, this gives a set-theoretic function $f : \mathbb{F}_p \to \mathbb{Z}_p$, where

$$\underbrace{z}_{=1+\cdots+1} \mapsto \underbrace{z}_{=1+\cdots+1},$$

and the sums of z ones are taken in the respective ring, and every element in $\mathbb{Z}_p$ can be written uniquely as

$$x = \sum_{i \geq 0} p^i f(x_i).$$

Furthermore, note that the residue field of $\mathbb{Z}_p$ is the coefficient set $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$.

We might ask if elements of $x \in \mathbb{Z}_p$ may be described by other choices of coefficients given by set functions $f : \mathbb{F}_p \to \mathbb{Z}_p$ such that every x can be written uniquely as $x = \sum_{i \geq 0} p^i f(x_i)$. The answer, of course, is yes, but some questions arise:

- Is there a canonical choice of representatives f?
- What are the relations for addition and multiplication in terms of the coefficients?
- Is something similar possible for more general rings?

As for a generalization of this construction, we consider so-called perfect rings R of characteristic p (meaning that the Frobenius $x \mapsto x^p$ is an isomorphism). We want to construct a ring $W(R)$, the ring of Witt vectors with coefficients in R, similarly to the construction of $\mathbb{Z}_p$ out of $\mathbb{F}_p$ above; particularly, we want that $W(R)/pW(R) = R$. The Witt vector construction then gives a so-called perfect p-ring $W(R)$. This construction turns out to be a functor

$$R \mapsto W(R)$$

from the category of perfect rings of characteristic p to the category of perfect p rings. Moreover, given a perfect ring R, the associated perfect p-ring $W(R)$ is unique up to isomorphism.

This construction also works for arbitrary rings, but in that case one looses the uniqueness of the ring $W(R)$. As we will only need this construction under the assumptions above, we will restrict ourselves to this case.

2.2. **Teichmüller representatives.**

Definition 2.1. Let A be a (commutative, unital) ring which is Hausdorff and complete for the p-adic topology, such that $R = A/pA$ is a perfect ring of characteristic p, and such that p is not a zero divisor in A. Then we call A a perfect p-ring.

The Witt vector construction takes rings which play the role of R and gives a perfect p-ring:

$$R \mapsto W(R) = A$$

Example 2.2. $A = \mathbb{Z}_p$ with $R = \mathbb{F}_p$.

Before constructing the Teichmüller lifts, we need a lemma.

Lemma 2.3. *Let A be a ring, and let $x, y \in A$ such that $x \equiv y \mod pA$. Then, $\forall i \geq 0$, $x^{p^i} \equiv y^{p^i} \mod p^{i+1}A$.*

Proof. By induction and the binomial theorem for the base case and the induction step. □

Theorem 2.4. *Let A be a perfect p-ring and $R = A/pA$. Then*

(i) *There exists a unique system of representatives $[\cdot] : R \to A$ which commutes with p-th powers, i.e. $[x^p] = [x]^p$.*
(ii) *Let $a \in A$. Then $a \in [R]$ iff for all n, a is a p-th power (in A).*
(iii) *The representatives are compatibile with multiplication: $\forall a, b \in R$, $[ab] = [a] \cdot [b]$.*

Proof. (ii) and (iii): follow from (i) and similar arguments. (Exercise.)

(i) : Let $x \in R$. We will construct $[x]$. Take an arbitrary lift $\widehat{x} \in A$ of x. Let $x^{(n)}$ be the p^n-th root of x in R (note that R is perfect, so the p^n-th root is unique), so $(x^{(n+1)})^p = x^{(n)}$. Take arbitrary lifts of $x^{(n)}$ as well, call them $\widehat{x^{(n)}}$.

Claim: The sequence $\left(\widehat{x^{(n)}}\right)^{p^n}$ converges in A to an element $[x]$ which depends only on x.

Apply the lemma above to $x = \widehat{x^{(n+1)}}^p, y = \widehat{x^{(n)}}$ to show that the sequence is a Cauchy sequence. The ring A is complete, which shows convergence. To show the independence of the lifts, if $\widehat{\widehat{\cdot}}$ is another lift, the apply the lemma to $x = \widehat{x^{(n)}}, y = \widehat{\widehat{x^{(n)}}}$.

Thus the given sequence converges and depends only on x. Denote the limit by

$$\lim_{n\to\infty} \left(\widehat{x^{(n)}}\right)^{p^n} = [x].$$

□

Definition 2.5. The element $[x] \in A$ is called the Teichmüller lift of $x \in R$.

Remark 2.6. If $a \in A$, then a has a unique representation of the form

$$a = \sum_{n\geq 0} p^n [a_n], \quad a_n \in R.$$

This representation gives a bijection (as sets)

$$A \to \prod_{n\geq 0} R.$$

2.3. **Example.** Let S be the closure of $\mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0}$ with respect to the p-adic topology. Here, the notation $X_i^{p^{-\infty}}$ means that we're adjoining all p^n-th roots of X_i. Then

(1) $S/pS = Z_p[...]/pZ_p[...] = \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0} = R$, a perfect field of characteristic p.
(2) S is a complete Hausdorff ring, since $\cap_{n\geq 0} p^n \mathbb{Z}_p[...] = 0$.
(3) p is not a zero divisor.

So we see that S is a perfect p-ring. Note that the Teichmüller lift of $X_i \in R$ is $[X_i] = X_i \in S$, and similarly for Y_i.

Now consider the elements $\sum_{n\geq 0} p^n X_n$ and $\sum_{n\geq 0} p^n Y_n$ in S. Their sum and product again are elements in S and by remark 2.6, there are elements $S_n, P_n \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0}$ such that

$$\sum_{n\geq 0} p^n X_n + \sum_{n\geq 0} p^n Y_n = \sum_{n\geq 0} p^n [S_n] \tag{2.1}$$

$$\sum_{n\geq 0} p^n X_n \cdot \sum_{n\geq 0} p^n Y_n = \sum_{n\geq 0} p^n [P_n]. \tag{2.2}$$

In fact, one can show that $S_n, P_n \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\leq n}$.

2.4. **Addition and multiplication in perfect p-rings.**

Proposition 2.7. *Let A be a perfect p-ring, $R = A/pA$, $\forall i : x_i, y_i \in R$. Then,*

$$\begin{aligned} \sum_{i\geq 0} p^i[x_i] + \sum_{i\geq 0} p^i[y_i] &= \sum_{i\geq 0} p^i[S_i(x_j, y_j)] \\ \sum_{i\geq 0} p^i[x_i] \cdot \sum_{i\geq 0} p^i[y_i] &= \sum_{i\geq 0} p^i[P_i(x_j, y_j)] \end{aligned}$$

Proof. Consider the homomorphism

$$\pi : \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0} \to A; X_i \mapsto [x_i], y_i \mapsto [y_i],$$

and then extend it by continuity to $\pi : S \to A$. Note that this is well-defined since the Teichmüller lifts commute with taking p-th powers.

Now apply π to equations (2.1), (2.2) and use that $[\cdot]$ commutes with π (resp. $\bar{\pi} : \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0} \to R$). □

Remark 2.8. By the proposition above, addition and multiplication in perfect p-rings are given by universal formulae and by the addition and multiplication structure of the residue ring R. Thus,

$$A \xrightarrow{\sim} \prod_{n\geq 0} R$$

as rings, where the ring structure on the right is determined by formulae (2.1), (2.2), and is not the usual component-wise structure.

2.5. **Construction of Witt vectors with coefficients in R.** Let R be a perfect ring of characteristic p, and let J be some set. Define

$$S_J := p\text{-adic closure of } \mathbb{Z}_p[X_j^{p^{-\infty}}]_{j\in J}, \qquad R_J := S_J/pS_J = \mathbb{F}_p[X_j^{p^{-\infty}}]_{j\in J}.$$

Similarly to S from above, S_J is a perfect p-ring.

Remark 2.9. If R is perfect of characteristic p, then R is the quotient of some R_J by an ideal I. Take e.g. $J = R$, then there is a surjection $R_J \twoheadrightarrow R, X_r \mapsto r$. Note that this surjection is well-defined since R is perfect of characteristic p.

Theorem 2.10. *Let R be a perfect ring of characteristic p. Then there is a unique perfect p-ring $W(R)$ such that*

$$W(R)/pW(R) = R.$$

If R' is another perfect ring of characteristic p and $f : R \to R'$ is a homomorphism, then there is a unique lifting

$$W(f) : W(R) \to W(R').$$

Proof. By the remark above, $R \xrightarrow{\sim} R_J/I$, where $I \lhd R_J$ is some ideal and J some set. Define

$$W(I) := \left\{ \sum_i p^i[x_i] \mid x_i \in I \right\} \lhd S_J.$$

Then define the Witt ring of R to be

$$W(R) := S_J/W(I).$$

Claim: This is a perfect p-ring with residue ring R.

- $W(R)/pW(R) \xrightarrow{\sim} S_J/(W(I)+pS_J) \xrightarrow{\sim} R_J/I \xrightarrow{\sim} R$, where the first two isomorphisms can be written down explicitely
- p is not a zero divisor in $W(R)$
- S_J is complete by definition, and $W(I)$ is closed is S_J (easy to verify), so $W(R)$ is complete.
- Since $\bigcap_{n\geq 0} p^n S_J = 0$,
$$\bigcap_{n\geq 0} (p^n S_J + W(I)) = W(I),$$
and therefore $W(I)$ is Hausdorff.

Now consider the isomorphism $W(R) = \prod_{n\geq 0} R$, where, as discussed above, the right side is determined by the formulae 2.1 and 2.2. Thus, the structure of the ring is fully determined by these formulae and the addition and multiplication in R, and so two such rings are canonically isomorphic. Thus, $W(R)$ is unique.

For a homomorphissm $f : R \to R'$, define $W(f) : W(R) \to W(R')$ by
$$W(f)\left(\sum_{i\geq 0} p^i[x_i]\right) := \sum_{i\geq 0} p^i[f(x_i)].$$
This is a necessary condition, since we want $W(f)$ to be a homomorphism and since the polynomials defining the addition satisfy
$$S_n(X_1,\dots,X_n,Y_1,\dots,Y_n) = X_n + Y_n + A_n(X_1,\dots,X_{n-1},Y_1,\dots,Y_{n-1}),$$
where A_n is a polynomial in $X_1,\dots,X_{n-1},Y_1,\dots,Y_{n-1}$. Furthermore, this is well-defined as a map. It remains to check that this indeed is a homomorphism:
$$\begin{aligned} W(f)\left(\sum_{i\geq 0} p^i[x_i] + \sum_{i\geq 0} p^i[y_i]\right) &= W(f)\left(\sum_{i\geq 0} p^i\left[s_i\left((x_j)_j,(y_j)_j\right)\right]\right) \\ &= \sum_{i\geq 0} p^i\left[f\left(S_i\left((x_j)_j,(y_j)_j\right)\right)\right] \\ &= \sum_{i\geq 0} p^i\left[S_i\left((f(x_j))_j,(f(y_j))_j\right)\right] \\ &= \sum_{i\geq 0} p^i\left[f(x_i)\right] + \sum_{i\geq 0} p^i\left[f(y_i)\right] \\ &= W(f)\left(\sum_{i\geq 0} p^i[x_i]\right) + W(f)\left(\sum_{i\geq 0} p^i[y_i]\right), \end{aligned}$$
and similarly for multiplication.

Thus, $W(f) : W(R) \to W(R')$ is a homomorphism and unique. □

Remark 2.11. The Frobenius $\varphi : R \to R$ extends to $W(\varphi) : W(R) \to W(R)$ and is often again denoted by φ.

Examples 2.12.

- $R = \mathbb{F}_p, \quad W(R) = \mathbb{Z}_p$,
- $R = \mathbb{F}_q, q = p^r, \quad W(R) = \mathbb{Z}_p[\zeta_{p-1}]$, where ζ_{p-1} is a primitive $(p-1)$-st root of unity,
- $R = R_J, \quad W(R) = S_J$ from above.

2.6. **Extending homomorphisms.** In a similar vein as the previous result, we'll show lifting for morphisms to rings complete in the p-adic topology:

Theorem 2.13. *Let A be a ring, complete for the p-adic topology. Let R be a perfect ring of characteristic p. Let $f : R \to A/pA$. Then there exists a unique $W(f) : W(R) \to A$.*

Proof. Let $x \in R$, and take any set-theoretic lifting $\widehat{f} : R \to A$. Let $x^{(n)}$ be the (unique, since R is perfect) p^n-th root of x.

Claim: The sequence $\left(\widehat{f}(x^{(n)})^{p^n}\right)_{n\geq 0}$ converges in A.

The proof is similar to the one of the construction of the Teichmüller representatives: Consider $y_n = f(x^{(n)}), \widehat{y_n} = \widehat{f}(x^{(n)})$. Then $\widehat{y_{n+1}}^p \equiv \widehat{y_n} \mod pA$ and by the lemma above, $\widehat{y_{n+1}}^{p^{n+1}} \equiv \widehat{y_n}^{p^n} \mod p^{n+1}A$. Therefore, the sequence converges.

For $x \in R$, define the desired map by

$$W(f)([x]) := \lim_{n\to\infty} \widehat{f}(x^{(n)})^{p^n}.$$

Then,

$$\widehat{f}(x^{(n)})^{p^n} \equiv W(f)([x]) \mod p^{n+1}A. \tag{2.3}$$

For any element $\sum_{i\geq 0} p^i[x_i] \in W(R)$, define

$$W(f)\left(\sum_{i\geq 0} p^i[x_i]\right) = \sum_{\imath\geq 0} p^i W(f)([x_i]).$$

This conditions are necessary and thus, it remains to show that this defines a homomorphism.

Since f is a homomorphism and R is perfect, $f\left((xy)^{(n)}\right)^{p^n} = f(x^{(n)})^{p^n} \cdot f(y^{(n)})^{p^n}$ and therefore,

$$W(f)\left([x]\cdot[y]\right) = W(f)([x]) \cdot W(f)([y]).$$

For arbitrary elements $\sum_{i\geq 0} p^i[x_i], \sum_{i\geq 0} p^i[y_i] \in W(R)$, the multiplicativity follows directly from the above, the definition, and since f is a homomorphism.

Additivity is more involved, and we will show it modulo p^n for every n. Recall that in S, the p-adic closure of $\mathbb{Z}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0}$,

$$(X_0 + pX_1 + \cdots + p^n X_n) + (Y_0 + pY_1 + \cdots + Y_n) \equiv [S_0] + p[S_1] + \cdots + p^n[S_n] \mod p^{n+1}S,$$

where $[S_i]$ is short for $[S_i\left((X_j)_{j\geq 0}, (Y_j)_{j\geq 0}\right)]$.

Denote the p^n-th root of $S_i\left((X_j)_{j\geq 0}, (Y_j)_{j\geq 0}\right) \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_i^{p^{-\infty}}]_{i\geq 0}$ by $S_i^{(n)}$, and a lift of this root by $\widehat{S_i^{(n)}} \in S$. Then,

$$\left(\widehat{S_i^{(n)}}\right)^{p^n} \xrightarrow{n\to\infty} [S_i],$$

and

$$\left(\widehat{S_i^{(n)}}\right)^{p^n} \equiv [S_i] \mod p^{n+1}S. \tag{2.4}$$

Therefore,

$$(X_0+pX_1+\cdots+p^nX_n)+(Y_0+pY_1+\cdots+Y_n) \equiv (\widehat{S_0^{(n)}})^{p^n}+p(\widehat{S_1^{(n)}})^{p^n}+\cdots+p^n(\widehat{S_n^{(n)}})^{p^n} \mod p^{n+1}S.$$

Let $x_0,\ldots,x_n,y_o,\ldots y_n \in R$. Similarly to the proof of the addition and multiplication formulae, using a suitable homomorphism, we apply this formula to $X_i = \widehat{f}(x_i^{(n)})^{p^n}$ and $Y_i = \widehat{f}(y_i^{(n)})^{p^n}$. Then,

$$\begin{aligned}\widehat{f}(x_0^{(n)})^{p^n} + p\cdot\widehat{f}(x_1^{(n)})^{p^n} + \cdots + p^n\cdot\widehat{f}(x_n^{(n)})^{p^n} + \widehat{f}(y_0^{(n)})^{p^n} + p\cdot\widehat{f}(y_1^{(n)})^{p^n} + \cdots + p^n\cdot\widehat{f}(y_n^{(n)})^{p^n}\\ \equiv \widehat{S_0^{(n)}}(\widehat{f}(x_i^{(n)})^{p^n},\widehat{f}(y_i^{(n)})^{p^n})^{p^n} + \cdots + p^n\cdot\widehat{S_n^{(n)}}(\widehat{f}(x_i^{(n)})^{p^n},\widehat{f}(y_i^{(n)})^{p^n})^{p^n} \mod p^{n+1}A\end{aligned}$$

Plugging (2.3) into this equation, we get

$$\begin{aligned}\sum_{i\geq 0} p^iW(f)([x_i]) + \sum_{i\geq 0} p^iW(f)([y_i]) &\equiv \sum_{i\geq 0} p^i\cdot\widehat{S_i^{(n)}}\Big((\widehat{f}(x_j^{(n)}))_j,(\widehat{f}(y_j^{(n)}))_j\Big)^{p^n} \mod p^{n+1}A\\ &\overset{(2.4)}{\equiv} \sum_{i\geq 0} p^i\left[S_i\Big((f(x_j^{(n)})^{p^n})_j,(f(y_j^{(n)})^{p^n})_j\Big)\right] \mod p^{n+1}A\\ &\overset{f\,hom.}{\equiv} \sum_{i\geq 0} p^i\cdot[f(S_i((x_j)_j,(y_j)_j))] \mod p^{n+1}A\\ &\equiv \sum_{i\geq 0} p^i\cdot W(f)([S_i((x_j)_j,(y_j)_j)]) \mod p^{n+1}A,\end{aligned}$$

Thus, for every $n\geq 0$,

$$W(f)\left(\sum_{i\geq 0}p^i[x_i]\right)+W(f)\left(\sum_{i\geq 0}p^i[y_i]\right)\equiv W(f)\left(\sum_{i\geq 0}p^i[x_i]+\sum_{i\geq 0}p^i[y_i]\right) \mod p^{n+1}A,$$

and thus, $W(f)$ is additive. □

We now give an application of this extension of homomorphisms which will be needed in later chapters.

Definition 2.14. Let R be a ring of characteristic p. Define the perfection of R by

$$\mathrm{Perf}(R) := \varprojlim R,$$

where the projective limit is taken with respect to the Frobenius $x\mapsto x^p$.

Corollary 2.15. *Let A be a ring, complete for the p-adic topology and let $x=(x_0,x_1,\ldots)\in \mathrm{Perf}(A/pA)$. For every i, choose a lift $\widehat{x}_i\in A$. Then $(\widehat{x}_i^{p^i})_{i\geq 0}$ converges in A to an element $x^{(0)}$ which only depends on x. Set $f:\mathrm{Perf}(A/pA)\to A$, $f(x)=x^{(0)}$. Then the map*

$$\theta: W(Perf(A/pA))\to A, \quad \sum_i p^i[x_i]\mapsto \sum_i p^ix_i^{(0)}$$

is a ring homomorphism.

Proof. Apply the theorem to the composition of f with the projection map $A\to A/pA$. □

Remark 2.16. The notation in $x^{(0)}$ differs from the notation of $x^{(n)}$ as the p^n-th roots of x from before. The notation used in this theorem is consistent with the notation used commonly in the literature.

Proposition 2.17. *The map θ is surjective iff the Frobenius morphism of A/pA is surjective.*

The corollary and the proposition will be applied later to the ring $A = \mathcal{O}_{\mathbb{C}_p}$. Furthermore, we will use the following notation.

Definition 2.18. The rings $\widetilde{\mathbb{E}}^+$ and $\widetilde{\mathbb{A}}^+$ are defined to be

$$\widetilde{\mathbb{E}}^+ := \operatorname{Perf}(\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}), \qquad \widetilde{\mathbb{A}}^+ := W(\widetilde{\mathbb{E}}^+).$$

By the corollary above, we get a homomorphism

$$\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}, \quad \sum_i p^i[x_i] \mapsto \sum_i p^i x_i^{(0)}$$

which is surjective by the proposition above.

2.7. **Witt vectors over valued rings.** Let R be complete for a valuation $\nu : R \to \mathbb{R} \cup \{\infty\}$.

Definition 2.19. The weak topology on $W(R)$ is the topology of "component-wise" convergence: if

$$a_n = \sum_{k=0}^{\infty} p^k [a_{n,k}]$$

Then $a_n \to 0$ iff $\forall k : \quad \nu(a_{n,k}) \to +\infty$ as $n \to \infty$.

In other words, considering the ring isomorphism $W(R) \xrightarrow{\sim} \prod_{n\geq 0} R$, then the weak topology on $W(R)$ is the topology on $W(R)$ induced by the product topology on $\prod_{n\geq 0} R$ where every component R is endowed with the topology coming from the valuation.

Thus, by construction, in this topology,

$$W(R) \xrightarrow{\sim} \prod_{n\geq 0} R$$

as topological rings. Moreover, because R is complete for the valuation ν, by this isomorphism of topological rings, $W(R)$ is complete for this topology:

Theorem 2.20. *$W(R)$ is complete in the weak topology.*

3. Cyclotomic extensions and the cohomology of $\mathbb{C}_p$

Lars Kühne on Monday, 19th of July, 2010.

To begin, some notation:

- $F := \mathbb{Q}_p, F_n := \mathbb{Q}_(\zeta_p)$
- $\zeta_{p^n} = p^n$-the root of unity.
- K/F finite extension.
- $K_n := \mathbb{Q}K(\zeta_{p^n})$
- $F_\infty := \cup_i F_i$, $K_\infty = \cup K_i$.
- $G_K := \operatorname{Gal}(F^{alg}/K)$

- $H_K := \mathrm{Gal}(F^{alg}/K_\infty)$.
- $G_K/H_K = \mathrm{Gal}(K_\infty/K)$.

3.1. **Basic facts.**

Proposition 3.1.

$$e_{F_n/F} = p^{n-1}(p-1) = [F_n : F], \qquad f_{F_n/F} = 1$$

Proposition 3.2.

$$\mathcal{O}_{F_n} = \mathcal{O}_F[\zeta_{p^n}] = \mathbb{Z}_p[\zeta_{p^n}]$$

We may define the cyclotomoic character χ on the Galois group of a cyclotomoic extension of the p-adic numbers.

Definition 3.3.

$$\mathrm{Gal}(F_n/F) \xrightarrow{\sim}_{\chi_n} (\mathbb{Z}/p^n\mathbb{Z})^*$$

Then take the inverse limit over n to define the character on $\mathrm{Gal}(F_\infty/F)$.

Some statement about $\zeta_{p^n}^G$ equals something.

The character is a continuous, open map with kernel:

$$\ker(\chi) = \mathrm{Gal}(F^{alg}/F_\infty) = H_F$$

So much for the extensions of F^{alg} over F_n and F_∞. The point of this talk is to deal with the more complex case of the extensions of $F^a lg$ over K_n or K_∞. First, a few observations:

- In general, $\mathcal{O}_{K_n} \neq \mathcal{O}_K[\zeta_{p^n}]$.
- For $n \gg 0$,
$$f_{K_{n+1}/K_n} = \overline{K_{n+1}} = \overline{K_n \cdot F_{n+1}} = \overline{K_n {\cdot} F_{n+1}} = \overline{K_n}.$$
- For $n \gg 0$, $\overline{K}_\infty = \overline{K}_n$.
- For $n \gg 0$, $[K_\infty : F_\infty] = ... = [K_n : F_n]$ and $[K_{n+1} : K_n] = [F_{n+1} : F_n] = p$.
- $\mathrm{Gal}(K_\infty/F_\infty) \xrightarrow{\sim} \mathrm{Gal}(K_n/F_n)$.

The tower of all K_n's behaves somewhat like the tower of all F_n's. Checking the difference between the differents of K_n and F_n shows they are not to far apart.

Theorem 3.4. *Retain the notation of the talk. Let $\mathfrak{d}_{K_n/F_n}$ be the different of K_n over F_n. Then the sequence $p^n \nu_p(\mathfrak{d}_{K_n/F_n})$ is bounded.*

Theorem 3.5. *Retain the notation of the talk. Then*

$$tr_{\ K_\infty/F_\infty}(\mathfrak{m}_{K_\infty}) = \mathfrak{m}_{F_\infty}.$$

Corollary 3.6. *For all $\delta > 0$, there exists a lower bound N such that for every $n \geq N$,*

$$K_n = \mathcal{O}_{F_n} e_1 \oplus \cdots \oplus \mathcal{O}_{F_n} e_d$$

for some e_i in some prescribed field. Then $\nu_p(e_i^*) \geq \delta$.

Theorem 3.7. *For all $\delta > 0$, there exists a lower bound N such that for every $n \geq N$, the following holds. Let $x \in \mathcal{O}_{K_{n+1}}$ and let $\sigma \in Gal(K_{n+1}/K_n)$. Then*

$$\nu_p(x^\sigma - x) \geq (\frac{1}{p-1} - \delta.$$

Furthermore,

$$\nu_p(N_{K_{n+1}/K_n}x - x^p) \geq \frac{1}{p-1} - \delta.$$

Corollary 3.8. *For such an N, let $I = \{x \in K_\infty \mid \nu_p(x) \geq \frac{1}{p-1} - \delta\}$. Then the following map is well-defined and surjective:*

$$\mathcal{O}_{K_{n+1}}/(I \cap \mathcal{O}_{K_{n+1}}) \to \mathcal{O}_{K_n}/(I \cap \mathcal{O}_k), \qquad \overline{x} \mapsto \overline{x^p} = \overline{N_{K_{n+1}/K_n}(x)}$$

Proof. This is well-defined. Surjectivity takes more work. □

3.2. Tate's normalized traces.

For $n \geq 1$, the morphism,

$$R_n : F_\infty \to F_n, \quad x \mapsto \frac{1}{p^k}\text{tr}\ _{F_{n+1}/F_n}(x),$$

is well-defined.

- R_n is in fact continuous (in p-adic topology). Thus $R_n : \widehat{F_\infty} \to F_n$.
- The limit $\lim_{n\to\infty} R_n(x) = x$.
- Then R_n satisfies the following inqueality evaluated at integers (from where?) π_n

$$R_n(\pi_n^j \mathcal{O}_{F_\infty} \leq \pi_n^j R_n(\mathcal{O}_{F_\infty}) \leq (\pi_n^j)$$

- We can calculate:

$$R_n : \zeta^j_{p_{n+k}} \mapsto \begin{cases} 0 & p^k \nmid j \\ \zeta^j_{p^{n+k}} & p^k \mid j \end{cases}$$

The R_n map is defined on the K_n's as well. We use the representation of elements of K_∞ as $x = \sum_{j=1}^d \text{tr}\ _{K_n/F_n}(xe_i)e_i^* = \sum_i x_i e_i^*$

$$R_n : K_\infty \to K_n, \qquad R_n : x \mapsto \sum_{j=1}^d R_n(x_i)e_i^*$$

for $e_i^* \in K_n$.

Returning to the F_n's and $R_n : \widehat{F_\infty} \to F_n$, let $X_n := \ker(R_n)$. $\widehat{F_\infty} = F_n \oplus X_n$ Then

$$(1_{\widehat{F_\infty}} - \alpha\gamma_n) : X_n \to X_n$$

is bijective.

3.3. The cohomology. $H^0(G_K)$

Theorem 3.9. *Let $\psi : G_K \to \mathbb{Z}_p^* \subset Aut(\mathbb{Q}_p)$ be a character such that $\psi(H_K) = 1$. Then let $\mathbb{Q}_p(\psi)$ be the one-dimensional representation, and $\mathbb{C}_p(\psi)$ be it's extension. Then the zeroth cohomology of $\mathbb{C}_p(\psi)$ is given by*

$$H^0(G_K, \mathbb{C}_p(\psi)) = \begin{cases} 0 & \psi \text{ infinite} \\ K & \psi \text{ finite} \end{cases}$$

The first cohomology has dimension over K given by,

$$\dim_K H^1(G_K, \mathbb{C}_p(\psi)) = \begin{cases} 0 & \psi \text{ infinite} \\ 1 & \psi \text{ finite} \end{cases}$$

.

3.4. **Admissible representations.**

Definition 3.10. Consider a representation $\rho : G_K \to Aut(V)$ of G_K on V, a $\mathbb{Q}_p$-vector space. We call it B-admissible (for some sort of B's which have G_K actions, of which $\mathbb{C}_p$ is an example), if

$$\dim_{B^{G_K}} (B \otimes_{\mathbb{Q}_p} V)^{G_K} = \dim_{\mathbb{Q}_p} V$$

Notice that $\mathbb{C}_p^{G_K} = \widehat{K} = K$.

4. The field E-tilde

Hiep Pham on Tuesday, the 20th of July, 2010.

Let A be a ring of characteristic p. As we saw earlier,

$$\mathrm{Perf}(A) := \varprojlim A = \{(x_0, x_1, \ldots) \mid x_{i+1}^p = x_i\},$$

where the limit is taken over the system $x \mapsto x^p$.

Lemma 4.1. *If $\mathcal{O}$ is a p-adically separable (i.e., Hausdorff) complete ring, then, the multiplicative map*

$$\begin{aligned} \phi : Perf(\mathcal{O}) &\to Perf(\mathcal{O}/p\mathcal{O}) \\ x = (x_n)_n &\mapsto (x_n \ mod \ p\mathcal{O})_n \end{aligned}$$

is a bijection.

Proof. We define

$$\psi : \mathrm{Perf}(\mathcal{O}/p\mathcal{O}) \to \mathrm{Perf}(\mathcal{O})$$

for any n, choosing a lifting $\widehat{x_n} \in \mathcal{O}$ of x_n. Then

$$\exists x^{(n)} = \lim_{m\to\infty} \widehat{x_{n+m}}^{p^m} \in \mathcal{O}$$

since $(x^{(n+1)})^p = x^{(n)}$. If $x \in \mathrm{Perf}(\mathcal{O})$, then $x = (x_n)_n = (x^{(n)})$ for some $x_n \in \mathcal{O}/p\mathcal{O}$ and $x^{(n)} \in \mathcal{O}$.

It is easy to see that this map is the inverse of ϕ. $\square$

Note that $(xy)^{(n)} = x^{(n)}y^{(n)}$ and $(x+y)^{(n)} = \lim_{m\to\infty}(x^{n+m} + y^{n+m})^{p^m}$

Definition 4.2.

$$\widetilde{E}^+ = \mathrm{Perf}(\mathcal{O}_{\mathbb{C}_p})/p\mathcal{O}_{\mathbb{C}_p})$$

Proposition 4.3. $(\nu_E, \widetilde{E}^+)$ *is a complete valuation ring.*

We can easily check that

- $\nu_E(xy) = \nu_E(x)\nu_E(y)$.
- $\nu_E(x) = 0 \iff \nu_p(x^{(n)}) = \infty \iff x^{(n)} = 0 \iff x = 0$.
- $\nu_E(x+y) \geq \min\{\nu_E(x), \nu_E(y)\}$.

Assume $x, y \neq 0$. Then $x^{(n)}, y^{(n)} \neq 0$ and $\nu_E(x) = \nu_p(x^{(n)}) = p^n \nu_p(x^{(n)})$ for all n. From this it follows that there exists some $n \gg 0$ such that

$$\nu_p(x^{(n)}), \nu_p(y^{(n)}) < 1$$

since $(x^{(n)} + y)^{(n)} \cong x^{(n)} + y^{(n)} \mod p$, we have

$$\nu_p((x+y)^{(n)}) \geq \min\{\nu_p(x^{(n)}), \nu_p(y^{(n)}), 1\}$$

By the previous equation, this means $\nu_p((x+y)^{(n)}) \geq \min\{\nu_p(x^{(n)}), \nu_p(y^{(n)})\}$.

If $(x_n)_n$ is a Cauchy sequence in $\widetilde{E}^+$, then $(x_n^{(0)})_n$ is a Cauchy sequency in $\mathcal{O}_{\mathbb{C}_p}$. But by completeness, this limit converges to $x^{(0)} = \lim_{n\to\infty} x_n^{(0)} \in \mathcal{O}_{\mathbb{C}_p}$. Thus we see

$$\lim_{n\to\infty} x_n = x = (x^{(0)}, x^{(1)}, \ldots) \in \widetilde{E}^+$$

so $(\nu_E, \widetilde{E}^+)$ is complete.

Definition 4.4.

$$\widetilde{E} := \mathrm{Frac}(\widetilde{E}^+)$$

It has a valuation inherited from $\widetilde{E}^+$.

Then

$$\mathcal{O}_{\widetilde{E}} = \{x \in \widetilde{E} \mid \nu_E(x) \geq 0\} = \{x \in \widetilde{E} \mid x^{(0)} \in \mathcal{O}_{\mathbb{C}_p} = \widetilde{E}^+$$

The maximal ideal is $\mathfrak{m}_{\widetilde{E}^+} = \{x \in \widetilde{E} \mid \nu_E(x) > 0\}$.

Consider the morphisms

$$\psi : \widetilde{E}^+ \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p} \twoheadrightarrow \widetilde{\mathbb{F}_p}$$

The first morphism is called θ_0, a special case of the family of maps

$$\theta_n : \widetilde{E}^+ \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}, \qquad (x_n) \mapsto x_n$$

The second is called ϕ and is the quotient map, and the composition, as marked, is ψ.

The kernel of ψ is $\mathfrak{m}_{\widetilde{E}^+}$, and

$$\widetilde{E}^+/\mathfrak{m}_{\widetilde{E}^+} \cong \widetilde{\mathbb{F}_p}.$$

$$\epsilon = (\epsilon^{(n)}) = (1, \zeta_p, \zeta_{p^2}, \ldots)$$

Let $\pi := \epsilon - 1$. Then π has valuation $\nu_E(\pi) = \frac{p}{p-1}$ and $\widetilde{E} = \widetilde{E}^+\left[\frac{1}{\pi}\right]$.

Theorem 4.5. *$\widetilde{E}$ is algebraically closed.*

Proof. It suffices to show that for any $P(X) \in \widetilde{E}^+[x]$ which is separable and homogeneous, then $P(x)$ has a root in $\widetilde{E}^+$.

Then there exist $U_0, V_0 \in \widetilde{E}[X]$ such that $U_0 P + V_0 P' = 1$. Then there exists some m and some $u \in \widetilde{E}^+$ such that

$$U = u^m U_0, \qquad V = u^m V_0 \in \widetilde{E}_+[X]$$

Then $\nu_E(u) = 1$, where $u = (p^{(n)})$ and $p^{(n)} = p$.

For fixed n, there exists $x \in \widetilde{E}^+$ such that $\nu(p(x)) \geq p^n \iff \theta_n(P(X)) = 0$.

Then $u_0 = 2m+1$ and $(x_n) \subset \widetilde{E}^+$ so we can contenct

$$\nu_E(x_{n+1} - x_n) \geq n - m, \qquad P(x_n) \in u^n \widetilde{E}^+$$

So we see there exists $\widetilde{x} = \lim_{n\to\infty} x_n \in \widetilde{E}^+$ which is a root of P. □

4.1. **The field H_L.** Some more definitions:

- $H_L := \ker(\chi : G_L \to \mathbb{Z}_p^*) = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{alg}}/L_\infty)$
- $\widetilde{E}_L^+ := \{x \in \widetilde{E}^+ \mid x_i \in \mathcal{O}_{L_\infty}/p\mathcal{O}_{L_\infty} \forall i \geq 0\}$.
- $\widetilde{E}_L := \widetilde{E}_L^+ \left[\frac{1}{n}\right]$.

These satisfy a number of properties:

- $\widetilde{E}_L = \widetilde{E}^{H_L}$.
- If $K/\mathbb{Q}_p$ is a finit eextension of fields, then
$$\overline{\widetilde{E_K}} = \cup_{L/K} \widetilde{E_L}$$
where the union is over finite extensions.
- $\widetilde{E_K}$ is dense in $\widetilde{E}$.
- $\mathrm{Gal}(\overline{\widetilde{E_K}}/\widetilde{E_K}) = H_K$.
- $E_{\mathbb{Q}_p}^+ = \mathbb{F}_p[[\pi]]$.
- $E_{\mathbb{Q}_p} = \mathbb{F}_p((\pi))$.
- $E = E_{\mathbb{Q}_p}^{\mathrm{sep}} \subset \widetilde{E}$.

We conclude with a theorem.

Theorem 4.6. *The morphism*

$$H_{\mathbb{Q}_p} \to Gal(E/E\mathbb{Q}_p)$$

is an isomorphism.

5. Some A's and B's and motivation

Thomas Preu on Tuesday, the 20th of July, 2010.

5.1. **Motivation.** Classical Hodge theory is the following situation. Let X be a projective $\mathbb{C}$ manifold. Then

$$\oplus_{p+q=i} H^q_{\mathrm{Sh}}(X, \Omega^p) \cong H^i_{\mathrm{dR}}(X) \cong \mathbb{C} \otimes_{\mathbb{Z}} H^i_{\mathrm{sing}}(X, \mathbb{Z})$$

If X is a smooth projective variety over L, a finite extension of $\mathbb{Q}$, then

$$\mathbb{C} \otimes_L \oplus_{p+q=i} H^q_{\mathrm{Sh}}(X, \Omega^p) \cong \mathbb{C} \otimes_L H^i_{dR} \cong \mathbb{C} \otimes_{\mathbb{Z}} H^i_{\mathrm{Sing}}(X_{\mathrm{an}}, \mathbb{Z})$$

This can be generalized from projective to proper varieties, and Deligne generalized it even more over fields of characteristic 0 using mixed Hodge structures.

The interprestation is $\mathbb{C}$ is a field of periods linking algebra with topology and geometry.

The questions is, can we replace the X_{a} by a non-archimedian (p-adic) analogue? If so, we'll need another ring of periods. Here are some theorems along those lines. If $L/\mathbb{Q}_p$ is finite, then

Theorem 5.1 (Tate, Faltings)**.** *For X a smooth proper variety over a finite extension $L/\mathbb{Q}_p$:*

$$B_{HT} \otimes_L grH^*_{dR}(X/L) \cong B_{HT} \otimes_{\mathbb{Q}_p} H^*_{\acute{e}t}(X \times_L \overline{L}, \mathbb{Q}_p)$$

Theorem 5.2 (Fontaine, Faltings)**.** *For X a smooth proper variety over a finite extension $L/\mathbb{Q}_p$:*

$$B_{dR} \otimes_L H^*_{dR}(X/L) \cong B_{dR} \otimes_{\mathbb{Q}_p} H^*_{\acute{e}t}(X \times_L \overline{L}, \mathbb{Q}_p)$$

Theorem 5.3 (Fontaine, Jansen, Faltings)**.** *For X a smooth proper variety over a finite extension $L/\mathbb{Q}_p$ of semistable reduction:*

$$B_{st} \otimes_L H^*_{dR}(X/L) \cong B_{st} \otimes_{\mathbb{Q}_p} H^*_{\acute{e}t}(X \times_L \overline{L}, \mathbb{Q}_p)$$

Theorem 5.4 (Fontaine, Faltings)**.** *For X a smooth proper variety over a finite extension $L/\mathbb{Q}_p$ of good reduction:*

$$B_{cris} \otimes_L H^*_{dR}(X/L) \cong B_{cris} \otimes_{\mathbb{Q}_p} H^*_{\acute{e}t}(X \times_L \overline{L}, \mathbb{Q}_p)$$

We saw yesterday that things were trivialized by the action of $\mathbb{C}_p$ on the cohomology, so we need to move to other rings to catch all the geometric information.

He presented a dictionary.

5.2. **Definitions.** We want to have rings to compute cohomology. The rings we saw last lecture were of characteristic p, but we'd like to tensor with fields of characteristic 0. Thanks to Claudia, we have the Witt vector construction which makes a ring of characteristic 0 out of a ring of characteristic p.

Since $\widetilde{E}^+$ is a perfect ring of characteristic $p > 0$, we may use the Witt vector construction to make the following definitions.

Definition 5.5. We define four new rings.

- $\widetilde{A}^+ := W(\widetilde{E}^+) \subset \widetilde{B}^+ := \widetilde{A}^+[1/p]$.
- $\widetilde{A} := W(\widetilde{E}) \subset \widetilde{B} := \widetilde{A}[1/p]$.

One would like to have an un-tilde, un-plus version. What one would wish to do is like this: $A := W(E) \subset B := A[1/p]$, but it doesn't work since E is no longer perfect. Of course, we may lift the Frobenius morphism ϕ to the Witt vectors $W(\phi)$. Also, we may lift Galois actions by g acts by $W(g)$ for $g \in \mathrm{Gal}_{\mathbb{Q}_p}$. So the action onn $\widetilde{A}^+$ and $\widetilde{A}$ lifts by functoriality to $\widetilde{B}^+$ and $\widetilde{B}$.

We have $\theta : \widetilde{E}^+ \to \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$. Sine $\mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$ is complete, the Witt vector construction gives

$$\Theta : \widetilde{A}^+ = W(\widetilde{E}^+) \twoheadrightarrow \mathcal{O}_{\mathbb{C}_p}$$

which is surjective because the Frobenius morphism is surjective. This localizes to $\Theta : \widetilde{B}^+ \twoheadrightarrow \mathbb{C}_p$.

Definition 5.6. Define

$$\pi := [\epsilon] - 1 \in \widetilde{A}^+, \qquad \mathrm{res}(\pi) = \epsilon - 1$$

Definition 5.7. Define

$$A_{\mathbb{Q}_p} := \widehat{Z_p((\pi))} \subset \widetilde{A}, \qquad B_{\mathbb{Q}_p} := A_{\mathbb{Q}_p}[1/p]$$

also

$$B := B_{\mathbb{Q}_p}^{\mathrm{unr}}, \qquad A := B \cap \widetilde{A}$$

There is a morphism

$$\begin{array}{rlc} \Theta : L \otimes_{L_0} \widetilde{B}^+ & \twoheadrightarrow L \otimes_{L_0} \mathbb{C}_p \to & \mathbb{C}_p \\ \sum_{i \geq k} \pi_L^i [x_i] & \mapsto & \sum_{i \geq k} \pi_L^i x_i^{(0)} \end{array}$$

where π_L is a uniformizer for L.

6. Galois invariant decomposition of B-dr

Aleksander Momot on Tuesday, the 20th of July, 2010.

Definition 6.1. Define B_{dR}^+ to be the formal completion of $\widetilde{B}^+$ with respect to $\ker \theta$.

The object of the talk is the following theorem.

Theorem 6.2. *Let $B_{dR} = Frac(B_{dR}^+)$. Then there is a "nice" element ϵ such that*

$$B_{dR} = B_{dR}^+[1/t] = \oplus_{n\geq 0} t^{-n} B_{dR}^+$$

gives a Galois invariant decomposition in the following sense: if $K/\mathbb{Q}_p$ is a finite extension, $g(t) = \chi(g)\cdot t$ for all $g \in G_K$, and there is a common diagram.

$$\begin{array}{ccc}
\widetilde{B}^+ = \widetilde{A}^+\left[\frac{1}{p}\right] & \xrightarrow{\ \widetilde{\theta}\ } & \mathcal{O}_{\mathbb{C}_p}\left[\frac{1}{p}\right] = \mathbb{C}_p\ . \\
\uparrow & & \uparrow \\
x, y \in \widetilde{A}^+ = W(\widetilde{E}^+) & \xrightarrow{W(\theta)=\theta} & \mathcal{O}_{\mathbb{C}_p} \\
{\scriptstyle mod\ p}\downarrow & & {\scriptstyle mod\ p}\downarrow \\
\overline{x}, \overline{y} \in \widetilde{E}^+ = Perf(R) & \longrightarrow & \mathcal{O}_{\mathbb{C}_p}/(p) = R
\end{array}$$

(Note that $Perf(R) \cong \lim_{\leftarrow} R$ where the limit is taken over $x \mapsto x^p$.) We call the morphism from $\widetilde{E}^+ = Perf(R)$ to $\mathcal{O}_{\mathbb{C}_p}$ by the name f.

Lemma 6.3. *Let $x \in \widetilde{A}^+$ such that $\nu_E(\overline{x}) = 1$. Then*

$$\ker \theta = x\widetilde{A}^+, \quad \theta(x) = 0.$$

Proof. Let $y \in \ker \theta$. Then by the diagram, $\nu_E(\overline{y}) \geq 1$. $\overline{y}/\overline{x} \in \widetilde{E}^+$, so

$$\ker \theta \cong x\widetilde{A}^+ \mod p$$

Then

$$\Theta_n x\widetilde{A}^+ + p^k \ker \theta = \ker \theta$$

We'll show this by induction. For $n = 1$, it's true. Let $n > 1$. p-adic completeness yields the claim. □

The conclusion of this discussion is $(\widetilde{B}^+, \ker \widetilde{\Theta})$ is a DVR, hence $(B_{\mathrm{dR}}^+, \ker \overline{\Theta})$ is a DVR.

We move on to an oberservation. $(\widetilde{B}^+, \ker \widetilde{\Theta})$ admits discrete valuation ν such that

$$\nu(t) = 1 \iff (t) = \ker \widetilde{\Theta}$$

Then $[\epsilon] - 1$ is a generator of $\ker \widetilde{\Theta}$ in B_{dR}^+. Construct t as follows. We'd like that

$$\text{“}t = \log([\epsilon])\text{''}$$

where $\epsilon = (\zeta_0, \zeta_1, \zeta_2, \ldots)$. We have

$$\nu_E([\epsilon] - 1) = \frac{p}{p-1}$$

Definition 6.4. Define exponentiation by $a \in \mathbb{Z}_p$ by

$$[\epsilon] := (1 + ([\epsilon] - 1))^a = \sum_{k \geq 0} \binom{a}{k} ([\epsilon] - 1)^k$$

where $\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}$ this will live in B_{dR}^+.

Definition 6.5. Define the logarithm by

$$\log(x+1) := \sum_{k \geq 1} (-1)^{k-1} \frac{x^k}{k} \in \mathbb{Q}_p^{\mathrm{alg}}[[x]]$$

It converges iff each term go to zero, i.e., arbitrarily high powers of $[\epsilon]$ divide it. Evaluated at $[\epsilon] - 1$, this gives

$$\log(([\epsilon] - 1) + 1) = \sum_{k \geq 1} (-1)^{k-1} \frac{([\epsilon] - 1)^k}{k} \in B_{\mathrm{dR}}^+.$$

Let $g \in G_K$. Then $g([\epsilon]) = [\epsilon]^{\chi(g)}$. Why? Whenever $\chi(g) \in \mathbb{Z}$ we know. Then extend the result by continuity to all $g \in G_K$.

Define $t = \log(([\epsilon] - 1) + 1)$ and

$$g(t) = \log(([\epsilon]^{p^n} - 1)^{\chi(g)} + 1) = \log([\epsilon]^{\chi(g)}) = \chi(g) \cdot \log(([\epsilon] - 1) = 1).$$

Now we have our decomposition, and

$$B_{\mathrm{dR}} = \cup_{n \geq 0} t^{-n} B_{\mathrm{dR}}^+$$

is a G_K-invariant filtration.

Theorem 6.6. *Then*

$$y \in \frac{1}{t^n} B_{dR}^+ \setminus B_{dR}$$

Proof. There is an exact sequence

$$0 \to tB_{\mathrm{dR}}^+ \to B_{\mathrm{dR}}^+ \to \mathbb{C}_p \to 0.$$

Tensor with "$t^n \in \mathbb{Q}_p$".

$$0 \to t^{n+1} B_{\mathrm{dR}}^+ \hookrightarrow t^n B_{\mathrm{dR}}^+ \to \mathbb{C}_p(\chi^n) \to 0$$

Case $n > 0$: Then

$$H^0(G_K, \mathbb{C}_p(\chi^n)) = 0.$$

Also,

$$(t^{n+1} B_{\mathrm{dR}}^+)^{G_K} = (t^n B_{\mathrm{dR}}^+)^{G_K}.$$

By induction,

$$(tB_{\mathrm{dR}}^+)^{G_K} \subset \cap_{n \geq 1} t^n B_{\mathrm{dR}}^+ = 0$$

Case $n < 0$: It is essentially the same argument.

Case $n = 0$: There is a left exact sequence

$$0 \to (tB_{\mathrm{dR}}^+)^{G_K} \to (B_{\mathrm{dR}}^+)^{G_K} \to (\mathbb{C}_p(\chi^n))^{G_K}$$

We know the first term is 0. We conclude $\mathbb{C}_p^{G_K} = K$.

If you accept that

$$K \subset B_{\text{dR}}^{+}.$$

is a G-equivariant way. Then the last arrow

$$(B_{\text{dR}}^{+})^{G_K} \to \mathbb{C}_p^{G_K}$$

is an isomorphism.

Writing $\mathbb{Q}_p^{\text{alg}}$ as a union of finite extensions, we arrive at the previously mentioned diagram. □

7. de Rham representations

Jun Yu on Tuesday, the 20th of July, 2010.

We're in the following setting: $B_{\text{dR}} \supset K \supset \mathbb{Q}_p$ and $B_{\text{dR}} \supset \overline{Q}_p$. Then G_K acts on B_{dR} with $B_{\text{dR}}^{G_K} = K$.

For a p-adic representation V of G_K, let

$$D_{dR}(V) = (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

which is a $B_{\text{dR}}^{G_K} = K$ vector space. There is an injective morphism

$$\alpha_{dR}(V) : B_{\text{dR}} \otimes_K D_{\text{dR}}(V) \to B_{\text{dR}} \otimes_{\mathbb{Q}_p} V.$$

Now we're ready for a definition.

Definition 7.1. A p-adic representation V of G_K is called de Rham if

$$\dim_K B_{\text{dR}}(V) = \dim_{\mathbb{Q}_p} V$$

Equivalently, it is a B_{dR}-admissible representation, or again, a representation such that $\alpha_{dR}(V)$ is an isomorphism.

We then define the category of de Rham representations $\text{Rep}_{\mathbb{Q}_p}^{\text{dR}}(G_K)$. We'll also define Filt_{dR} as the category of finite dimensional $\mathbb{C}$-vector spaces equipped with an action of...didn't get it.

Index of Z with

(1) $\text{Fil}^{i+1} D \subset \text{Fil}^i D$
(2) $\text{Fil}^i D = 0$ for $i \gg 0$ and $\text{Fil}^i D = D$ for $i \ll 0$.

The filtration category Fil_K is a tensor category.

$$\text{Fil}^i (D_1 \otimes D_2) = \sum_{i_1+i_2=i} \text{Fil}^{i_1} D_i \otimes \text{Fil}^{i_2} D_i$$

In teh case $D = K$,

$$\text{Fil}^i = K, i \leq 0, 0, i > 0$$

Let V be a p-adic representation of G_K, and $D_{\text{dR}}(V)$ is a filtered K-vector space. Then

$$\begin{aligned} \text{Fil}^i D_{\text{dR}}(V) &= \text{Fil}^i B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ \text{Fil}^i B_{\text{dR}} &= t^i B_{\text{dR}}^{+} \end{aligned}$$

Theorem 7.2. *The functor*

$$D_{dR} : Rep_{\mathbb{Q}_p}^{dR} \to Fil_K$$

is an exact, faithful and tensorial functor.

Proof. Later if we have time. □

7.1. **Hodge-Tate representation.** $B_{HT} = \oplus_{i\in\mathbb{Z}}\mathbb{C}_p(i)$ where $\mathbb{C}_p(i) := C_p$??

Then
$$g \cdot (\chi \cdot t^i) = \chi^q(g)g(\chi)t^i$$
where χ^q is the cyclotomic chracter. $\chi \cdot t^i ycdott^j = x \cdot yt^{i+j}$.

Proposition 7.3. *de Rham-admissible implies something else is Hodge-Tate-admissible, and*
$$\dim_K D_{dR}(V) = \sum_{i\in\mathbb{Z}} \dim_K gr^i D_{dR}(V)$$
where $gr^i D_{dR}(V) = t^i B^+_{dR}/t^{i+1}B^+_{dR})^{G_K}$. *Then*
$$\sum_{i\in\mathbb{Z}} t^i B^+_{dR}/t^{i+1}B_{dR} = B_{HT}$$
where the terms are equal to $\mathbb{C}_p(i)$.

If char $E = 0$ and X/E is a projective smooth algebraic variety with de Rham complex
$$\Omega_{X/E} : \mathcal{O}_{X/E} \to \Omega^1_{X/E} \to \cdots$$
Then define the de Rham cohomology group by
$$H^n_{dR}(X/E) := H^m(\Omega_{X/E}), m \in \mathbb{N}$$
where the second term is hypercohomology.

Theorem 7.4 (Falting-Tsuj). *Let* $E = K/\mathbb{Q}_p$ *and* $V = H^m_{\acute{e}t}(X_{\overline{k}}, \mathbb{Q}_p)$ *is a de Rham representation and there is a canonical isomorphism of filtered* K*-vector spaces*
$$D_{dR}(H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\sim} H^m_{dR}(X/K)$$

Theorem 7.5. *Also*
$$B_{dR} \otimes_{\mathbb{Q}_p} H^m_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \xrightarrow{\sim} B_{dR} \otimes_K H^m_{dR}(X/K)$$
gives rise to the notion of p*-adic Hodge structure.*

Conjecture 7.6 (Fontaine-Muiur). *Geometric representations are exactly the representations coming from algebraic geometry by the above construction.*

Definition 7.7. Given V, if $G_{\mathbb{Q}}$ is geometric.

(1) It is unramified away from finitely many p.
(2) It is de Rham at $p = t$ for all primes p.

8. Bmax inside BdR

Joseph Ayoub on Tuesday, the 20th of July, 2010.

8.1. **Setting.**

- $\widetilde{E}^+ = \mathrm{Perf}(G_{\mathbb{C}_p}/p)$.
- $\theta : \widetilde{A}^+ = W(\widetilde{E}^+) \to G_{\mathbb{C}_p}$. The kernel is $\ker(\theta) = (\xi)$ where $\xi = [\widetilde{p}] - p$ for $\widetilde{p} \in \widetilde{E}^+$ such that $\widetilde{p}^{(0)} = p$.
- $\widetilde{B}^+ = \widetilde{A}^+[1/p]$.

In some sense, the goal of this ring $B_{\max}$ is to distinguish $\mathbb{Q}_p$ Galois representations coming from smooth, projective varieties over $\mathbb{Q}_p$ with good reduction. Also B_{cris} seems to do the same thing, but $B_{\max}$ is better behaved. Things converge better in $B_{\max}$.

Then

$$B_{\text{dR}} = \widetilde{B}^{+}//(\xi) \supset B_{\max}$$

Definition 8.1. We have $P = \{\sum_{i=0}^{t}(\xi)^i/p^i$ and $(\xi) \subset \widetilde{A}^{+}$. Then define

$$\begin{aligned} A^{+}_{\max} &= P//(p) \\ B^{+}_{\max} &= A^{+}_{\max}[1/p] \end{aligned}$$

Definition 8.2. Let

$$t = \log[\epsilon] = \sum_{i\geq 0} \frac{(-1)^i(1-[\epsilon])^i}{i} \in A^{+}_{\max}$$

then

$$B_{\max} = B^{+}_{\max}[1/t]$$

There is a second defintion of $B_{\max}$. We introduce a valuation on $\widetilde{B}^{+}$ as follows:

$$\nu(f = \sum_{n > -\infty} p^n[x_n]) := \inf\{\nu_{\widetilde{E}^{+}}(x_n) + n \mid n\}$$

Lemma 8.3. *This is indeed a valuation.*

Definition 8.4. Let $B^{+}_{\max}$ be the completion of $\widetilde{B}^{+}$ with respect to this valuation.

Proposition 8.5. *These two constructions give the same object, i.e., P is the valuation ring of ν and*

$$P//p \cong (\text{ completion of the valuation ring of } \nu).$$

Proof. We have

$$\left\langle \left(\frac{\xi}{p}\right)^i \right\rangle_{i\geq 1} = P$$

as $\widetilde{A}^{+}$-modules.

Consider $p^{-i}\cdot[x_i]$ and $\nu_{\widetilde{f}}(x_i) = x_i = \tilde{p}_{\tilde{p}^i}\cdot u$ where $u \in \widetilde{E}^{+}$. Then $p^{-i}\cdot[x_i] = [u]\cdot[\tilde{p}]^i/p^i$. □

8.2. **Properties.**

- There is a ϕ (Frobenius morphism?) on $B_{\max}$ (not on B_{dR}).
- From the construction with also have a Galois action of $G_{\mathbb{Q}_p}$ on $B_{\max}$.

$K/\mathbb{Q}_p$ a finite extension with K_0 the maximal unramified extension. Then consider $B_{\max}^{G_K} \subset K$.

There is a Frobenius on $B_{\max}$, and we can use it to show that the fixed field of G_K is unramified. The fixed points of the action on

Proposition 8.6. *There is a morphism,*

$$K \otimes_{K_0} B_{max} \to B_{dR},$$

and it is injective.

The proof is rather involved.

9. More on Bmax

Daniel Haase on Tuesday, the 20th of July, 2010.

I will continue to examine the properties of B_{max}. On the ring B_{dR}, we have a filtration given by

$$\mathrm{Fil}^n B_{\mathrm{dR}} = t^n \cdot B_{\mathrm{dR}}^+$$

This will be used in the classification of the de Rham representations. There is a counterpart of this for B_{max}. We can define

$$B^{\phi=\lambda} = \{x \in B \mid \phi(x) = \lambda \cdot x\}$$

We'll find out in a bit what's allowed to play the role of the λ. Let's consider intersections:

$$B_{\mathrm{max}}^{\phi=\lambda} \cap \mathrm{Fil}^n B_{\mathrm{dR}} = ?$$

This will be either 0 or $\mathbb{Q}_p$ for the appropriate filtration.

9.1. Intermediate results.

Proposition 9.1 (14.1.2 in Berger). *For A_{max}^+*

$$\sum_{i=0}^{\infty} \sigma_j \left(\frac{[\tilde{p}]}{p}\right)^j$$

Lemma 9.2. *Then*

$$\widetilde{B}_{rig}^+ = \cap_{n\geq 1} \phi^n (B_{max}^+)$$

Proposition 9.3 (14.1.3 in Berger). *For all $n \geq 1$,*

$$(\widetilde{B}_{rig}^+)^{\phi=1} = \mathbb{Q}_p, \qquad (\widetilde{B}_{rig}^+)^{\phi=p^{-n}} = \{0\}.$$

Proposition 9.4 (19.2.7 in Berger). *Let $M \in (B_{max}^+)^{m\nmid n}, X \in (\widetilde{B}_{rig}^+)^{m\nmid n}, Y \in (\widetilde{B}_{rig}^+)^{m\nmid n}$. Then*

$$M = X \cdot \phi(M) \cdot Y$$

Computes of M in $\widetilde{B}_{rig}^+$.

Proposition 9.5 (16.1.2 in Berger). *If $Y \in \widetilde{B}_{rig}^+, \phi^n(y) \in t \cdot B_{dR}^+$ for all $n \in \mathbb{Z}$ then $y \in t \cdot B_{dR}^+$.*

Proposition 9.6 (6.2.7 in Berger). *The map $(1-\phi) : V \twoheadrightarrow V$ is surjective, so for all $\lambda_0 \in V$, there exists a $\mu \in V$ such that $\lambda_0 = n/\phi(\mu)$.*

9.2. A proposition with proof.

Proposition 9.7 (14.1.3 in Berger).

$$(\widetilde{B}_{rig}^+)^{\phi=1} = \mathbb{Q}_p, \qquad (\widetilde{B}_{rig}^+)^{\phi=p^{-m}} = \{0\}$$

Proof. It suffices to show that $A_{\mathrm{max}}^{\phi=1} = \mathbb{Z}_p$ acts by p-th poewrs on $(B_{\mathrm{max}}^+)^{\phi=1} = \mathbb{Q}_p = (\widetilde{B}_{\mathrm{rig}}^+)^{\phi=1}$.

Let $y \in A_{\mathrm{max}}^+$ be a fixed point of ϕ. Where

$$y = \sum_{j=0}^{\infty} u_j \left(\frac{[\tilde{p}]}{p}\right)^j$$

for some $u_j \in \widetilde{E}^+$. Then

$$y = \phi^*(y) = \sum_{j=0}^{\infty} \phi^*(u_j) \left(\frac{[\tilde{p}^{p^n}]}{p} \right)^j = \sum_{j=0}^{\infty} \phi(u_j) p^{\frac{m}{j(p^n-1)}} \cdot \left(\frac{[\tilde{p}]}{p} \right)^{jp^n}$$

So

$$y \in \widetilde{A}^+ + p^m A^+_{\max}$$

for all $m \gg 0$. Note that $\widetilde{A}^+$ is closed. So $y =? + p^n b_n$ where $? \to y$ and $p^n b_n \to 0$. Thus $y \in \widetilde{A}^+$. Also, $(\widetilde{A}^+)^{\phi=1} = \mathbb{Z}_p$. We conclude the first part of the result.

We have integers coming out $(\widetilde{A}^+)^{\phi=1} = \mathbb{Z}_p$ so we cannot have negative powers of p and we conclude the second part of the result. □

9.3. **Filtration property.**

Proposition 9.8. *We have*

$$\begin{aligned} B^{\phi=1}_{max} \cap Fil^0 B_{dR} &= \mathbb{Q}_p \\ B^{\phi=1}_{max} \cap Fil^1 B_{dR} &= \{0\} \end{aligned}$$

Proof. Let $y \in B_{\max} = B^+_{\max}[1/t]$, so $y = \sum_{k=0}^m y_k \cdot t^{-k}$ for $y_k \in B^+_{\max}$. From the definition of the logarithm series, $\phi(t^n) = p \cdot t^{-n}$. Then

$$\phi(y) = \sum_{k=0}^{m} \phi(y_k) \cdot p^k \cdot t^{-k}$$

Write

$$\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = X \cdot \begin{pmatrix} \phi(y_0) \\ \vdots \\ \phi(y_n) \end{pmatrix} = X \cdot \phi(n)$$

for some matrix X. Thus $y_k \in \widetilde{B}^+_{\text{rig}}$. Now let $y \in \widetilde{B}^+_{\text{rig}}[1/t]$, so $y \in t^{-m}\widetilde{B}^+_{\text{rig}}$. Then

$$\phi(t^m \cdot y) = p^m \cdot t^m \cdot y \in t \cdot \widetilde{B}^+_{\text{rig}}$$

Since we know that y is in the Richard ring $\widetilde{B}^+_{\text{rig}}$, using the proposition 16.1.2, we conclude (there's a little to be clarified) that $y \in \widetilde{B}^+_{\text{rig}}$. □

9.4. **Final result.** Let $\lambda \in W(\overline{\mathbb{F}_p})$ and $n := \nu_p(\lambda)$. Then

$$B^{\phi=\lambda}_{\max} \cap \text{Fil}^{n+1} B_{\text{dR}} = \{0\}, \qquad B^{\phi=\lambda}_{\max} \cap \text{Fil}^{n} B_{\text{dR}} = t^n \cdot \mu \cdot \mathbb{Q}_p$$

Multiplication with μ maps $B^{\phi=\lambda}_{\max}$ to $B^{phi=p^n}_{\max}$ compatible with "Fil". Then $\mu : \lambda = \frac{p^n \cdot \mu}{\phi(\mu)}$, and $\lambda_0 = \lambda / p^n \in W(\overline{\mathbb{F}_p})^K$.

10. Formal groups

Mingxi Wang on Wednesday, 21st of July, 2010.

We restrict ourselves to formal gropus of dimension 1.

Definition 10.1. A commutative formal group $\mathcal{F}$ defined over $\mathcal{O}$ is represented by $F(X,Y) \in \mathcal{O}[[X,Y]]$ such that

(1) $F(x,y) \equiv x+y \pmod{\text{degree 2 terms}}$
(2) $F(x,y) = F(y,x)$
(3) $F(x,F(y,z)) = F(F(x,y),z)$

Formal group laws have the following properties:

- There exists a unique $i(x) \in \mathcal{O}[[x]]$ such that $F(x,i(x)) = 0$ for all x.
- $F(x,0) = x$ and $F(0,y) = y$.

Define $\mathbb{G}_a$ by $G_a(x,y) = x+y$, and $\mathbb{G}_m$ by $\mathbb{G}_m(x,y) = x+y+xy$.

Definition 10.2. If F and G are formal groups over $\mathcal{O}$, then a homomorphisms $f : F \to G$ from F to G is

$$f(x) \in \mathcal{O}[[x]], \qquad \text{with no constant term}$$

such that $G(f(x),f(y)) = f(F(x,y))$.

For example, given F over $\mathcal{O}$ a formal gropu, then for all $m \in \mathbb{Z}$, we define:

- $\widehat{0}(x) = 0$.
- $(\widehat{m+1})(x) = F(\widehat{m}(x),x)$.
- $(\widehat{m-1})(x) = F(\widehat{m}(x),i(x))$.

In this case, $\widehat{m} \in \operatorname{End}(F)$ and $\widehat{m}(x) \equiv mx$ modulo degree two terms.

Definition 10.3. Give formal groups F and G over $\mathcal{O}$ and a homomorphism $f : F \to G$, we say f is an isomorphism if there is another homomorphism $g : G \to F$ such that

$$g \circ f(x) = f \circ g(x) = x.$$

Lemma 10.4. *Let F and G be (one-dimensional) formal groups over $\mathcal{O}$ and $f : F \to G$. Then if $f'(0) \in \mathcal{O}^*$, f is an isomorphism.*

Definition 10.5. A formal $\mathcal{O}$-module is a formal group F over $\mathcal{O}$ with a morphism

$$\begin{aligned} \mathcal{O} &\to \operatorname{End}(F) \\ a &\mapsto \widehat{a}(x) \in \mathcal{O}[[x]] \text{ such that } \widehat{a}x \equiv ax \bmod \deg 2. \end{aligned}$$

For example, $\mathbb{G}_m$ is a formal $\mathbb{Z}_p$-module by the morphism

$$\begin{aligned} \mathbb{Z}_p &\to \operatorname{End}(\mathbb{G}_m) \\ a &\mapsto \widehat{a}(x) = (1+x)^a - 1 = \sum_{i=1}^{\infty} \binom{a}{i} x^i \end{aligned}$$

10.1. **Differentials.**

Definition 10.6. A differential over $\mathcal{O}$ is a w such that

$$w = f(t)dt, \qquad f(t) \in \mathcal{O}[[t]].$$

We call it an invariant differential of F over $\mathcal{O}$ if

$$w(x) = w \circ F(x,y),$$

which could also be written $f(F(x,y))F_x(x,y) = f(x)$. We call w a normalized invariant different if it is invariant and the constant term of f is 1.

Proposition 10.7. *For a formal group F over $\mathcal{O}$, there exists a unique normalized invariant differential. We'll denote it w_F.*

Proof. There is a $w = f(t)dt$ such that $f(F(x,y))F_x(x,y) = f(x)$. Let $x = 0$. This becomes

$$f(x)F_x(0,y) = 1$$

But $F_x(0,y) \equiv 1 \mod y$ so $f(y) = 1/F_x(0,y)$. Conversely, if $w(t) = dt/F_x(0,t)$, then reading the argument backwards gives the result. □

For example $w_{\mathbb{G}_a} = dt$ is the normalized invariant differential of $\mathbb{G}_a(x,y) = x + y$. Also,

$$w_{\mathbb{G}_m}(t) = \frac{dt}{1+t} = (1 - t + t^2 - + \cdots)dt$$

is the normalized invariant differential of $\mathbb{G}_m(x,y) = x + y + xy$.

Corollary 10.8. *Given a morphism $f : F \to G$ of formal groups over $\mathcal{O}$, we have $w_G(f) = f'(0)w_F$.*

Proof. Easy exercise. □

Corollary 10.9. *Given a formal group F over $\mathcal{O}$ and*

$$\widehat{p}(x) = ph(x) = g(x^p)$$

for some $h, g \in \mathcal{O}[[x]]$.

Proof. First note that $\widehat{p}'(0) = p$. Note $w_F = (1 + b_1 t + \cdots)dt = (1 + b_1\widehat{p}(t) + \cdots)d\widehat{p}(t)$. By the previous corollary,

$$(1 + a_1 t + \cdots)\widehat{p}(t)dt = w_F(\widehat{p}(t)) = pw_F(t).$$

Thus $\widehat{p}'(t) \equiv 0 \mod p$. □

10.2. **Logarithms.**

Definition 10.10. Given a formal group law F over $\mathcal{O}$ with char $\mathcal{O} = 0$, then a logarithm over $\mathcal{O}$ is any morphism

$$f : F \to \mathbb{G}_a/\mathcal{O}$$

We'll assume $f \neq 0$. We say f is nondegenerate if $f \equiv x(\text{ mod deg } 2)$.

Proposition 10.11. *A logarithm $f : F \to \mathbb{G}_a/\mathcal{O} \otimes \mathbb{Q}$ always exists.*

Proof. We want to get $\log_F : F \to \mathbb{G}_a$ such that $\log_F(x) + \log_F(y) = \log_F(F(x,y))$. Derivate this equation to get

$$\log_F'(x) = \log_F'(F(x,y))F_x(x,y).$$

Letting $x = 0$ gives $1 = \log_F'(y)F_X(0,y)$. Thus

$$\log_F'(y) = \frac{1}{F_X(0,y)} = \frac{w_F(y)}{dt}$$

So $\log_f(y) = \int w_f(y)$. □

Now for an example. Take $\mathbb{G}_m$ over $\mathbb{Q}$. Then

$$\begin{aligned} w_{\mathbb{G}_m} &= \frac{dt}{1+t} \\ \log_{\mathbb{G}_m}(t) &= \int \frac{dt}{1+t} = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots \end{aligned}$$

10.3. **de Rham cohomology.** We'll define the de Rham cohomology of formal groups (of dimension one) over $\mathcal{O}_K$ where K is a finite, unramified extension of $\mathbb{Q}_p$.

Definition 10.12. Let F be a formal group over $\mathcal{O}$ of characteristic p. We define the height of F, $h(F)$, to be the largest $h \in \mathbb{N}$ such that

$$\widehat{p}(x) = f(x^{p^h})$$

for some $f \in \mathcal{O}[[x]]$. More generally, if $char(\mathcal{O}/\mathfrak{m}) = p$, then define

$$h(F/\mathcal{O}) := h(\overline{F}/(\mathcal{O}/\mathfrak{m})).$$

Note: These equations should be interpreted in the residue field $\overline{K}$.

For all $w \in \mathcal{O}_K[[t]]dt$, let

$$I(w) = \int w \in K[[t]].$$

Definition 10.13. A w is exact if

$$I(w) \in K \otimes_{\mathcal{O}_k} \mathcal{O}_K[[t]]$$

w is of the second kind if

$$I(w)(F(x,y)) - I(w)(x) - I(w)(y) \in K \otimes_{\mathcal{O}_k} \mathcal{O}_K[[x,y]].$$

Define the first de Rham cohomology of F over $\mathcal{O}_K$ by

$$H^1_{\mathrm{dR}}(F/\mathcal{O}_K) = \frac{\text{second kind}}{\text{exact}}$$

Theorem 10.14. *The cohomology $H^1_{\mathrm{dR}}(F/\mathcal{O}_K)$ is a K-vector space of dimension $h(F)$.*

We show a correspondence between the first de Rham cohomology and the Tate module of the formal group.

11. Lubin-Tate modules and local class field theory

Philipp Habegger on Wednesday, the 21st of July, 2010.

The setting is K a local field, $\mathcal{O}_K$ is its ring of integers with uniformizer $\pi \in \mathcal{O}_K$ generating the unique prime ideal $\mathfrak{m}_K$. $\nu : K^* \twoheadrightarrow \mathbb{Z}$ is its (normalized, i.e., with range $\mathbb{Z}$) valuation. We'll fix an algebraic closure $\overline{K} \supset K$. We'll denote the absolute Galois group $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Finally, let $q = |\mathcal{O}_K/\mathfrak{m}_K|$.

Definition 11.1. A Lubin-Tate series with respect to π is an $f \in \mathcal{O}_K[[X]]$ such that

$$\begin{aligned} f &= \pi X \mod X^2 \mod \deg 2 \\ f &= X^q \mod \pi \\ f_\pi &:= \{\text{Lubin-Tate series w.r.t. } \pi\} \end{aligned}$$

For example

(1) $f = \pi X + X^q$ for any K.
(2) $K = \mathbb{Q}_p$, $f = (x+1)^p - 1 = pX + \cdots X^p$, for $\pi = p$.

Definition 11.2. A Lubin-Tate module with respect to π is a formal $\emptyset_K$-module $(F, \widehat{\cdot}\,)$ such that $\widehat{\pi} \in \mathcal{F}_\pi$.

For example, $\mathbb{G}_m$ over $\mathbb{Z}_p$ is a Lubin-Tate module with $F := XY + X + Y$, $\pi = p$ and $\widehat{p} = (x+1)^p - 1 \in \mathcal{F}_p$.

Lemma 11.3. *Let $f, g \in \mathcal{F}_\pi$, and let $L \in \mathcal{O}_K[X_0, \ldots, X_n]$ be a linear form. Then there exists a unique $F \in K[[X_1, \ldots, X_n]]$ such that*

(1) $F \equiv L \mod \deg 2$.
(2) $f(F(X_1, \ldots, X_n)) = F(g(X_1), \ldots, g(X_n))$.
(3) $F \in \mathcal{O}_K[[X_1, \ldots, X_n]]$.

Proof. By induction on r, where we'll construct a power series for each r, $F_r \in K[X_1, \ldots, X_n]$ such that $\deg F_r \leq r$ and

$$f(F_r(X_1, \ldots, X_n)) \equiv F(g(X_1), \ldots, g(X_n)) \mod \deg r+1$$

Define them as follows:

$$\begin{aligned} F_1 &= L \\ F_{r+1} &= F_r + \Delta_r \end{aligned}$$

where $\Delta_r \in \mathcal{O}_K[[X_1, \ldots, X_n]]$ and $\Delta_r = 0 \mod \deg r+1$. Define Δ_r by

$$\Delta_r = \frac{f(F_r(X_1, \ldots, X_n)) - F_r(g(X_1), \ldots, g(X_n))}{\pi(\pi^{r-1} - 1)}$$

We need to check that $\Delta_r \in \mathcal{O}_K[[X_1, \ldots, X_n]]$. But $\pi^{r-1} - 1 \in \mathcal{O}_K^*$, so

$$f(F_r(X_1, \ldots, X_n)) - F_r(g(X_1), \ldots g(X_n)) = F_r(X_1, \ldots, X_n)^q - F_r(X_1^q, \ldots, X_n^q) = 0 \mod \pi$$

and we have what we want. □

Definition 11.4. Let $f \in \mathcal{F}_\pi$. If $f = g$ and $L = X + Y$, then the previous lemma gives us a formal power series which we will write as $F_f \in \mathcal{O}_K[[X, Y]]$. If $f, g \in \mathcal{F}_\pi$ and $a \in \mathcal{O}_K$ and $L = a \cdot X$, then the lemma gives us a power series denoted by $\widehat{a}^{f,g} \in \mathcal{O}_K[[X]]$. If $f = g$, the f, g in the notation will often be surpressed.

Lemma 11.5. (1) *$f \in \mathcal{F}_\pi$ implies $(F_f, \widehat{\cdot}\,)$ is a Lubin-Tate module with respect to π.*
(2) *If $f, g \in \mathcal{F}_\pi$, then $\widehat{1}^{f,g}$ is an isomorphism $(F_f, \widehat{\cdot}^f\,) \xrightarrow{\sim} (F_g, \widehat{\cdot}^g\,)$.*

In particular, up to isomorphism, there's only one Lubin-Tate module for a fixed π. See note below for clarification.

Proof. Does $F_f(F_f(X, Y), Z) = F_f(X, F_f(Y, Z))$? Look at the linear term $(X + Y) + Z = X + (Y + Z)$. Then the uniqueness in the first lemma says the equality is true. □

Note that going from a Lubin-Tate module $(F, \widehat{\cdot}\,)$ with respect π, going to the power series $f = \widehat{\pi} \in \mathcal{F}_\pi$ and then through the construction back to a Lubin-Tate module $(F_f, \widehat{\cdot}^{\,f}\,)$ gives the original module (i.e., $F_f = F$).

Corollary 11.6. *Any Lubin-Tate module with respect to* π *is isomorphic to* $(F_f, \widehat{\cdot}\,)$ *where* $f = \pi X + X^q$.

Beginning with an $\mathcal{O}_K$-module $(F, \widehat{\cdot}\,)$, L/K a finite extension, then if $x, y \in \mathfrak{m}_L$, since the coefficients of F are integers, $F(x,y)$ converges. Thus $\widehat{a}(x)$ converges for each $a \in \mathcal{O}_K$.

$$\begin{array}{rcl} (X,Y) & \mapsto & F(X,Y) = X +_F Y \\ (a,X) & \mapsto & \widehat{a}(X) = a_F \cdot X \end{array}$$

determines an $\mathcal{O}_K$-module structure on $\mathfrak{m}_{\overline{K}}$.

Now for some new notation:

Definition 11.7. Let $f \in \mathcal{F}_\pi$, $g = f$ and $L = X + Y$. Then the first lemma gives a power series denoted

$$F_f \in \mathcal{O}_K[[X,Y]]$$

Let $f, g \in \mathcal{F}_\pi$, $a \in \mathcal{O}_K$ and $L = a \cdot X$. Then the lemma gives a power series denoted

$$\widehat{a}^{f,g} \in \mathcal{O}_K[[X]]$$

If $f = g$, the f, g will often be surpressed in the notation.

Lemma 11.8. *Let* $f \in \mathcal{F}_\pi$ *and* $n \in \mathbb{N}$. *Then*

1. $F_f[\pi^n]$ *is a free* $\mathcal{O}_K/\mathfrak{m}_K^n$*-module of rank 1.*
2. $F_f[\pi^n] \subset K^{sep}$ *and* G_K *acts on* $F_f[\pi^n]$.
3. $L_n := L_{\pi,n} := K(F_f[\pi^n])$, *is independent of* f. *It's an finite, abelian and totally ramified extension of* K *and there is an isomorphism*
$$Gal(L_n/K) \xrightarrow{\sim} Aut_{\mathcal{O}_K/\mathfrak{m}_K^n}(F_f[\pi^n]) \cong (\mathcal{O}_K/\mathfrak{m}_K^n)^*$$
4. $\pi \in N_{L_n/K}(L_n^*)$.

Proof. WLOG, assume $f = \pi X + X^q$ by Lemma 2, and $\lambda \in F_f[\pi^n]$, and $f^{\circ n}(\lambda) = 0$. Then

$$\frac{f^{\circ n}}{f^{\circ(n-1)}} = (f^{\circ(n-1)})^{q-1} + \pi$$

where the superscripts with the $\circ n$ denote composition with itself n times.

$$\begin{array}{rcl} q^n \geq |F_f[\pi^n]| & \Rightarrow & \mathcal{O}_K/(F_f \mathfrak{m}_K^n) \cdot \lambda = F_f[\pi^n] \\ & \Rightarrow & f^{\circ n} \text{ is separable } \Rightarrow \lambda \in K^{\text{sep}} \end{array}$$

Then

$$\mathrm{Gal}(K^{\text{sep}}/K) \to \mathrm{Aut}_{\mathcal{O}/\mathfrak{m}_K^n} I_f[\pi^n]$$

where the automorphisms are a commutative group. Then $N_{K_n/K}(\lambda) = \pi$ since $f^{(n)}(\lambda) = 0$. Thus

$$\nu(\pi) = [L_n : K] \cdot \nu(\lambda)$$

and we see L_n/K is totally ramified. □

Definition 11.9. If $f \in \mathcal{F}_\pi$, then we define

$$T_\pi(F_f) := \varprojlim_n F_f[\pi^n]$$

There are morphisms

$$\Pi_{F_f} : F_f[\pi^{n+1}] \to F_f[\pi^n]$$

free $\lim_{\longleftarrow} \mathcal{O}_K/\mathfrak{m}_K^n$-module of rank 1.

For example, $K = \mathbb{Q}_p$, $\pi = q = p$, $\widehat{p} = (1+x)^p - 1$. Then

$$F[p^n] = \{\zeta - 1 \mid \zeta^{p^n} = 1\}$$

11.1. **LCFT.**

Theorem 11.10. *Let L/K be a finite abelian extension. Then there exists a natural isomorphism*

$$r_{L/K} : Gal(L/K) \to K^*/N_{L/K}(L^*)$$

(1) *The map $L \mapsto N_{L/K}(L^*)$ is a bijection.*

$$\{\textit{finite abelian extensions of } K\} \to \{\textit{subgroups of } K^* \textit{ which are open and finite index}\}$$

(2) *L/K is a finite, abelian, unramified extension iff $N_{L/K}(L^*) \supset \mathcal{O}_K^*$.*

First, we define

$$K^* \twoheadrightarrow K^*/N_{L/K}(L^*) \to \mathrm{Gal}(L/K)$$

The first map is called β and the second map is called $r_{L/K}^{-1}$. The composition of these two maps is written

$$\left(\frac{\cdot}{L/K}\right)$$

and called the norm residue symbol.

Theorem 11.11 (Lubin-Tate)**.** *Let L/K and $\lambda \in F_f[\pi^n]$, $u \in \mathcal{O}_K^*$ a unit. Then*

$$\left(\frac{u}{L_{\pi,n}/K}\right)\lambda = \widehat{u^{-1}}(\lambda)$$

For example, $K = \mathbb{Q}_p$ and $f = (1+x)^p - 1$. Then

$$L_{\pi,n} = \mathcal{O}_p(\zeta|\zeta^{p^n} = 1)$$

Also

$$\left(\frac{u}{L_{\pi,n}/K}\right)(\zeta - 1) = \zeta^{u^{-1}} - 1 = \widehat{u^{-1}}(\zeta - 1)$$

Dwosk showed that

$$\left(\frac{u}{L_{\pi,n}/K}\right)\zeta = \zeta^{u^{-1}}$$

Corollary 11.12. *Let K be a local field, K^{ab} be the maximal abelian extension of K, K^{unr} the maximal unramified extension of K. Then*

$$K^{ab} = K^{unr} \cdot \cup_{n \gg 1} L_{\pi,n}.$$

Proof. Uses local class field theory and the Lubin-Tate construction. □

12. Semi-stable representations

Giovanni di Matteo on Thursday, the 22nd of July, 2010.

12.1. **Review.** We've defined a lot of things so far:

- $B_{\max}$ which has an action by the Galois group $G_{\mathbb{Q}_p}$ and a morphism $\phi : B_{\max} \to B_{\max}$.
- If $K/\mathbb{Q}_p$ is a finite extension, $B_{\max}^{G_K} = K_0 = K \cap \mathbb{Q}_p^{n,r}$. There is an injection $K \otimes_{K_0} B_{\max} \hookrightarrow B_{\mathrm{dR}}$, and a filtration $\mathrm{Fil} B_{\mathrm{dR}} \cap B_{\max}^{\phi=\lambda}$.
- $B_{\mathrm{st}} = B_{\max}[Y]$ and has an action of the Galois group $G_{\mathbb{Q}_p}$ by
$$\sigma : Y \mapsto Y + c(\sigma)t$$
thus $\sigma(p^{1/p^n}) = p^{1/p^n}(\epsilon^{(n)})^{c(\sigma)}$. Also
- $\phi(Y) = pY$.
- $N = -\frac{\partial}{\partial Y}$.
- $B_{\mathrm{st}}^{G_K} = K_0$ and $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ by
$$Y \mapsto \text{“}log(\widehat{p})\text{”} := -\sum_n \frac{(1-\frac{\widehat{p}}{p})^{n-1}}{n}$$

Note $\overline{\sigma}$ is the absolute Frobenius morphism.

12.2. **Introduction.**

Definition 12.1. Let V be a p-adic representation G_K is semi-stable if
$$B_{\mathrm{st}} \otimes V \cong B_{\mathrm{st}}^d$$
by a G_K-equivariant isomorphism.

Proposition 12.2. *Let V be a p-adic representation G_K. Then the morphism*
$$B_{st} \otimes_p (B_{st} \otimes V)^{G_K} \hookrightarrow B_{st} \otimes_{\mathbb{Q}_p} V$$
Is injective and respects the additional structure. We define
$$D_{st}(V) := (B_{st} \otimes V)^{G_K}$$
Also,
$$\dim_{K_0} D_{st}(V) \leq \dim_{\mathbb{Q}_p} V$$
Then the following are equivalent:

(1) *V is semi-stable.*
(2) *The above morphism is an isomorphism.*
(3) $\dim_{K_0} D_{st}(V) = \dim_{\mathbb{Q}_p} V$

Let V be a p-adic representation. Then $D_{\mathrm{st}}(V)$ and K_0 are finite dimensional vector spaces. The morphism ϕ is additive and injective with
$$\phi(\lambda x) = \overline{\sigma}(\lambda) \cdot \phi(x), \qquad \lambda \in K_0$$
for $\overline{\sigma} : K_0 \to K_0$.

Also, $N : D_{\mathrm{st}}(V) \to D_{\mathrm{st}}(V)$ is K_0-linear and nilpotent satisfying,
$$N\phi = p\phi N$$

There is a filtration $K \otimes_{K_0} D_{\rm st}(V) \hookrightarrow D_{\rm dr}(V) = (bdr \otimes V)^{G_K} \supset (t^k B_{\rm dR} \otimes V)^{G_K}$.

Definition 12.3. We say a p-adic vector space V is crystalline if

$$B_{\max} \otimes V \cong B_{\max}^d$$

In that case, we define,

$$\begin{aligned} D_{\rm cris}(V) &= (B_{\max} \otimes V)^{G_K} \\ D_{\rm cris}(V) &= D_{\rm st}(V)^{N=0} \end{aligned}$$

Note that $D_{\rm cris}(V)$ is ϕ-stable and there is an inclusion $K \otimes_{K_0} D_{\rm cris}(V) \hookrightarrow D_{\rm dR}(V)$.

Proposition 12.4. *If V is crystalline, then V is semi-stable.*

$$\dim_{Q_p} V = \dim_{K_0} D_{\rm cris}(V) \leq \dim_K D_{\rm st}(V) \leq \dim_{Q_p} V$$

Proposition 12.5. *Let $V = \mathbb{Q}_p(\eta)$ for some character eta : $G_K \to \mathbb{Z}_p^*$ mapping $\sigma \mapsto \eta_\sigma$. Then $\dim D_{st}(V) \leq 1$ implies $N = 0$. Also, $D_{cris}(V) = D_{st}(V)$.*

So in dimension 1, being crystalline is equivalent to being semi-stable.

Proposition 12.6. *$V = \mathbb{Q}_p(\eta)$ is crystalline iff $\eta = \chi^h \cdot \mu$ for some μ non-ramified, $h \in \mathbb{Z}$.*

Proof. There exists $b \in B_{\max}$ such that $\eta_\sigma \cdot \sigma(b) = b$ for all $\sigma \in G = G_{\mathbb{Q}_p}$. Then $B_{\max} \hookrightarrow B_{\rm dR}$ so we may view b as an element of $B_{\rm dR}$.

$b = t^{-h} \cdot b_0$ for some $h \in \mathbb{Z}$ and $b_0 \in B_{\rm dR}^+ - tB_{\rm dR}^+$. Then

$$\eta_\sigma = \frac{b}{\sigma(b)} = \frac{t^{-h} \cdot b_0}{\sigma(t^{-h} \cdot b_0)} = \frac{t^{-h} b_0}{\chi(\alpha)^{-h} t^{-h} \sigma(b_0)} = \chi^h(\sigma) \cdot \frac{b_0}{\sigma(b_0)}$$

Define $\eta' : \sigma \mapsto b_0/\sigma(b_0)$. Then η' is crystalline. The situation is as follow.

$$\begin{aligned} \eta' : G_{\mathbb{Q}_p} &\to \mathbb{Z}_p^* \\ G_{\mathbb{Q}_p} &\to G_{Q_p}^{\rm ab} \\ \eta' : G_{Q_p}^{\rm ab} &\to \mathbb{Z}_p^* \end{aligned}$$

By the local Weber? theorem we know that ever finite abelian extension is contained in a cyclotomic one. If we look at those roots which are relatively prime to p, they are unramified, and those divisible by p are totally ramified.

$$G_{Q_p}^{\rm ab} \cong {\rm Gal}(\mathbb{Q}_p^{\rm ab}/\mathbb{Q}_p) \cong {\rm Gal}(\mathbb{Q}_p^{\rm nr}/\mathbb{Q}_p) \times {\rm Gal}(\mathbb{Q}_{p??}/\mathbb{Q}_p) \cong \Gamma$$

Then $\eta' = \mu \cdot \tau$ with μ non-ramified and τ in Γ.

Case $\tau = 1$: $\tau = \eta' \cdot \mu^{-1}$.

Lemma 12.7. *μ^{-1} is non-ramified then there exists $z \in \widehat{\mathbb{Q}_p^{nr}}$ such that*

$$\mu^{-1}(\sigma) = \frac{z}{\sigma(z)}, \qquad \forall \sigma$$

Thus $Z(\sigma) = \frac{b_0}{\sigma(b_0)} \cdot \frac{z}{\sigma(z)}$ for $b_0 z \in B_{\max}$ but $b_0 z \notin tB^+_{\mathrm{dR}}$. Apply

$$\theta : B^+_{\mathrm{dR}} \to \mathbb{C}_p$$

To get $\theta(b_0 z) \in \mathbb{C}_p(\tau)^{G_{\mathbb{Q}_p}}$, which is nonzero. Since τ is of finite order, the Ax-Sen-Tate theorem gives that $\tau|_{G_{F_n}}$. So the totally ramified extension $F_n/\mathbb{Q}_p$ is trivial.

Now $b_0 z \in (B_{\max})^{G_{F_n}} = \mathbb{Q}_p$. So by the lemma, $\eta = \chi^h \cdot \mu$. □

Proposition 12.8. *The category of crystalline (rep. st) is stable by under the operations of taking subobject, quotient object, direct sum sum and tensor (alo duals?). Furthermore D_{cris} (resp. D_{st}) respects these operations.*

12.3. Filtered (ϕ, N)-modules.

Definition 12.9. Let $K/\mathbb{Q}_p$ be finite. Then a filtered (ϕ, N)-module over K is

- D a K_0-vector space
- $\phi : D \to D$ an additive, injective map with $\phi(\lambda x) = \overline{\sigma}(\lambda)\phi(x)$.
- $N : D \to D$ such that $N\phi = p\phi N$.

Definition 12.10. A filtration on $D_K := K \otimes_{K_0} D$ is a choice for all $i \in \mathbb{Z}$ of K-subspaces

$$\mathrm{Fil}^i D_K \leq D_K$$

such that $\mathrm{Fil}^{i+1} D_K \leq \mathrm{Fil}^i D_K$ and $\cup_{i \in \sigma} \mathrm{Fil}^i D_K = D_K$ and $\cap_i \mathrm{Fil}^i D_K = 0$.

Definition 12.11. A morphism $T : D \to D'$ of (ϕ, N)-modules is a K_0-linear map such that

- $NT = TN$
- $T\phi = \phi T$
- $T(\mathrm{Fil}^i D_K) \subset \mathrm{Fil}^i D'_K$

Then $D \otimes D'$ is a K_0-vector space and $D \otimes D'$ and $\phi \otimes \phi'$ and $N \otimes 1 + 1 \otimes N'$. Then

$$\mathrm{Fil}^i D_K \otimes D'_K = \sum_{u+v=i} \mathrm{Fil}^u D_u \otimes \mathrm{Fil}^v D_K.$$

Note that if D is finite dimensional and $E = (e_1, \ldots, e_d)$ is a K_0-basis of D, $A = (a_i j)$, and $\phi(e_j) = \sum i = 1^d a_{ij} e_i$. Also $E' = (e'_1, \ldots, e'_d)$, $Mat(\phi|E') := A' = (a'_{ij})$ Then

$$M : E \to E'$$

Then $G = \sum_{i=1}^d m_{ij} e'_i$ and $MA = A'\overline{\sigma}(M)$.

Proposition 12.12. *If D is finite dimensional, then it is nilpotent.*

Proof sketch. If we assume N is not nilpotent, since D is Aritnian as a K-algebra, there exists a $k > 0$ such that $N^k(D) = N^{k+1}(D) = N^{k+2}(D) = \cdots \neq 0$. Define $D' := N^k(D)$, which is stabler by N, N surjective.

Also stable by ϕ: $\chi = N^K(y)$ so $\phi(x) = \phi(x) = \phi N^k(g) = p^{-k} N^k(\phi(g))$ where $N^k(\phi(g)) \in D'$.

So $N\phi = p\phi N$ gives $NA = pA\overline{\sigma}(N)$. Then $\phi(\det(\cdot))$, and thus $0 = 1$, a contradiction! □

Proposition 12.13. *The functor*

$$D_{st} : \{\textit{semi-stable } p\textit{-adic repn of } G_K\} \to \{\textit{Filtered } (\phi, N)\textit{-modules}\}$$

is fully faithful.

Proof. Let V be semistable. Then

$$B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) \xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V.$$

So

$$\begin{aligned} N = 0 : B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) &\xrightarrow{\sim} B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V \\ \mathrm{Fil}^0 : B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) &\xrightarrow{\sim} \mathrm{Fil}^0(B_{\max}) \otimes_{\mathbb{Q}_p} V \\ \phi = 1 : B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) &\xrightarrow{\sim} \mathrm{Fil}^0(B_{\max})^{\phi=1} \otimes_{\mathbb{Q}_p} V \end{aligned}$$

But $\mathrm{Fil}^0(B_{\max})^{\phi=1} = \mathbb{Q}_p$.

Then $D = D_{\mathrm{st}}(V) \mapsto (B_{\mathrm{st}} \otimes D)^{N=0, \mathrm{Fil}^0, \phi=1}$ gives the quasi-inverse. □

Proposition 12.14. *Similarly, the functor*

$$D_{cris} : \{\textit{crystalline repns}\} \hookrightarrow \{\textit{Filtered } (\phi, N)\textit{-modules with } N = 0\}$$

is fully faithful.

Let D be a filtered (ϕ, N)-module. Then define $t_H(D) \in \mathbb{Z}$ to be its Hodge number and $t_N(D) \in \mathbb{Z}$ to be its Newton number. This will characterize the image $D_{\mathrm{st}}(\cdot)$.

Given a filtered ϕ-module of finite dimension, $\mathrm{Fil}^i D_K \leq D_K$ and

$$0 = \mathrm{Fil}^{i_1} D_K \leq \cdots \leq \mathrm{Fil}^{i_j} D_K \leq\neq \mathrm{Fil}^i D_k = D_K$$

Let $d_j := \dim_K \mathrm{gr}^{i_j} D_K$. The picture is as follows:

We see

$$t_H(\eta) = \sum_{j=1}^{n} i_j \dim \mathrm{gr}^{ij} D_K$$

Note that if D is of dimension 1, then $t_H(D)$ is the ???? $h \in \mathbb{Z}$ such that $\mathrm{gr}^h D_K \neq 0$. One can show if D of d

$$t_H(\wedge^d D) = t_H(D)$$

where $\dim \wedge^d D = 1$.

Now ϕ has a the slope decomposition over $\widehat{K_0^{\mathrm{nr}}}$ which then determines the Newton polygon and $t_N(D)$. If $d = \dim D$, then

$$t_N(D) = t_N(\wedge^d D)$$

If $D = K_0 \cdot e$ is of dimension 1, then $t_N(D) = \nu_p(\lambda)$ where $\phi(e) = \lambda(e)$.

Definition 12.15. A filtered (ϕ, N)-module D is weakly admissible if

(1) $t_H(D) = t_N(D)$
(2) for all $D' \leq D$, $t_H(D') < t_N(D')$. Or less than equal??

Proposition 12.16. *If V is semi-stable, then $D = D_{st}(V)$ is weakly admissible.*

Proof. First, "$t_H(D) = t_N(D)$". If V is semi-stable, then D is of dimension $d = \dim_{\mathbb{Q}_p} V$.

Since the category of semi-stable representations is closed under products and quotients, $\wedge^d V$ is semistable. Then

$$\wedge^d V = \chi^h \cdot \mu, \qquad \mu \text{ non-ramified}, h \in \mathbb{Z}$$

Let $t^{-h} b_0$ be a period. Since $b_0 \in \widehat{\mathbb{Q}_p^{\mathrm{nr}}}$, the produc with t^{-h} is in $t^{-h} B_{\mathrm{dR}}$. We have $t_H = -h$.

In dimension one, the Newton number was defined by how the Frobenius acts on a basis.

$$\phi(t^{-h}b_0) = p^{-h}t^{-h}\frac{\overline{\sigma}(b_0)}{b_0}b_0$$

so $d = p^{-h}\frac{\overline{\sigma}(b_0)}{b_0}$. Also $\nu_p(\lambda) = -h$.

Second, If $D' < D$ is a submodule of dimension $r < d$.

$$\wedge^r D' < \wedge^r D$$

Reset $D' < D$ where $\dim D' = 1$. N is nilpotent so $N = 0$

Now remember, $D' \subset D_{\mathrm{st}}(V)^N = D_{\mathrm{cris}}(V) \subset B_{\mathrm{st}} \otimes V$ and $D' \subset (B_{\max} \otimes V)^{\phi=\lambda}$.

At this point we just recall that if $n = \nu_p(\lambda)$, then $B_{\max}^{\phi=\lambda} \cap \mathrm{Fil}^{n+1} B_{\mathrm{dR}} = \{0\}$. Necessarily, $h = t_H(D') \leq t_N(D')$ otherwise the intersection would be 0.

So we've identified the essential image of the functor. □

Theorem 12.17 (Colmez, Fontaine)**.** *There is a functor*

$$D_{st} : \{\textit{semi-stable p-adic representations of } G_K\} \xrightarrow{\sim} \{\textit{weakly admissible filtered } (\phi, N)\textit{-modules}\}$$

which is an equivalence with sub-equivalence

$$\{\textit{crystalline representations}\} \xrightarrow{\sim} \{N = 0\}$$

13. Summary of (much of) the things covered this week

Sergey Gorchinskiy on Thursday, the 22th of July, 2010.

`http://www.umpa.ens-lyon.fr/~lberger/barcelone/BergerBarcelone.pdf`

Give a non-trivial example of a high-dimensional crystalline representation. Where should we look for an example. Perhaps look at the Tate module of an elliptic curve. But we won't do that. Yesterday, we saw Lubin-Tate modules, and that's the representation we want to discuss.

13.1. **Notation.** Let's fix some notation:

- $K/\mathbb{Q}_p$ a local field.
- $d := [K : \mathbb{Q}_p]$ and the residue field of $\mathcal{O}_K$ is $\mathbb{F}_{p^f}$.
- $G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$ and $\pi \in K$ unramified.
- $\widetilde{A}_K^+ := \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} \widetilde{A}^+$ and $\widetilde{B}_K^+ := K \otimes_{K_0} \widetilde{B}^+$.
- An element of $\widetilde{B}_K^+$ is a Laurent series $\sum_{i > -\infty} [x_i]\pi^i$ for some $x_i \in \widetilde{E}^+$.
- There is a morphism $\widetilde{A}_K^+ \to \widetilde{E}^+$ given by reducing modulo π.
- $\theta_K : \widetilde{B}_K^+ \to \mathbb{C}_p$ is $\theta_K := (K \hookrightarrow \mathbb{C}_p) \otimes \theta$.
- G_K acts on $\widetilde{B}_K^+$ and $\phi_K := id \otimes \phi^f$.
- $\widetilde{A}_{\max,K}^+ := \mathcal{O}_K \otimes_{\mathcal{O}_{K_0}} A_{\max}^+$ and $\widetilde{B}_{\max,K}^+ = K \otimes_{K_0} B_{\max}^+$.
- $\widetilde{B}_{\max,K}^+$ is a V-completion of $\widetilde{B}_K^+$.

Local class field theory gives

$$\begin{array}{ccccccc} 1 \longrightarrow & \mathcal{O}_K^* & & K^* & \longrightarrow & \mathbb{Z} & . \\ & \downarrow \cong & & \downarrow r_K & & & \\ 1 \longrightarrow & H & \longrightarrow & G_K^{\mathrm{ab}} := G_K/\overline{[G_K : G_K]} & \longrightarrow & \widehat{\mathbb{Z}} = \mathrm{Gal}(K^{\mathrm{nr}}/K) & \end{array}$$

Now consider the composition

$$G_K \longrightarrow G_K^{\mathrm{ab}} \longrightarrow \mathcal{O}_K \subset GL(K/\mathbb{Q}_p) \cong GL_d(\mathbb{Q}_p)\ .$$

Definition 13.1. Denote the representation of G_K on K, given by χ_π, as $K(1)$.

Question 13.2. We will answer three questions.

- Is $K(1)$ B-admissible for some B? (e.g., $B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{st}}, B_{\max}$)
- If yes, find $D_B(K)$.
- If yes, find the transcendence degree of the periods of $K(1)$ in B (over $\mathbb{Q}_p^{\mathrm{alg}}$).

$$V \otimes_{\mathbb{Q}_p} B \cong D_B(V) \otimes_{K(K_0)} B$$

an isomorphism of G_K-modules.

We will answer these questions using the following strategy.

(1) We would like to work with $K(1)$. Until now, $K(1)$ is given by some hidden Artin map, see the first diagram. We will give a more explicit interpretation of $K(1)$ in arithmetic terms, namely, in terms of Lubin-Tate formal group laws.
(2) Then we will look at $K = \mathbb{Q}_p$. $K(1) = \mathbb{Q}_p(1)$ is cyclic character. Call the period $t \in B_{\max}$.
(3) Then interpret the period t in terms of Lubin-Tate.
(4) Do the same for any $K/\mathbb{Q}_p$ finite extension, getting a period $t_\pi \in B_{\max}$.
(5) Then calculate $D_{B_{\max}}(K(1))$. At this point will we have answered first two questions.
(6) Then we will give a statement about period of p-adic G_K-representations. Apply this to our case and get explicitly the transcendence degree of the periods.

13.2. **Arithmetic interpretation of $K(1)$.** Let $f(X) \in \mathcal{F}_\pi$ be a Lubin-Tate series, $F(X,Y)$ a Lubin-Tate formal group law. Let $\mathcal{O}_K \hookrightarrow \mathrm{End}(F)$ be the mapping $a \mapsto \widehat{a}$, and $T(F) = \lim_{\longleftarrow} F[\pi^n]$ where $F[\pi^n] \subset \mathfrak{m}_{K^{\mathrm{alg}}}$.

Local class field theory tells us that ever $\sigma \in G_K$ acts on $T(F)$ by $\widehat{\chi_\pi(G)}$. Then $K(1) \cong T(F) \otimes_{\mathcal{O}_K} K$.

Remark: $T_p(F) = \lim_{\longleftarrow} F[p^n] \cong T(F)$. The height of the Lubin-Tate group law is $h(F) = d$. Then

$$\pi^{e(K/\mathbb{Q}_p)} = pu, \quad u \in cO_K^*$$

and

$$h(F) = h(\widehat{p}) = h(\widehat{\pi}^e) = e \cdot h(\widehat{\pi}) = e \cdot f = d$$

13.3. **The case $K = \mathbb{Q}_p$.** Let's take $K = \mathbb{Q}_p$ with uniformizer $\pi = p$ and $f = (X+1)^p - 1$ with formal group law $F = \mathbb{G}_m = X + Y + XY$. Then

$$T(F) = \mathbb{Z}_p(1)$$

the cyclotomic character. Also $\chi_\pi = \chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^*$.

Let's test it for admissibility:

$$(\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} B_{\max})^{G_{\mathbb{Q}_p}} = (\mathbb{Q}_p \cdot t^{-1}) = D_{\text{cris}}(\mathbb{Q}_p(1))$$

The e be a generator of $\mathbb{Q}_p(1)$ over $\mathbb{Q}_p$. Then for all $\sigma = G_K$, $\sigma(t) = \chi(\sigma)t$.

Moreover,

- $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots) \in \widetilde{E}^+$
- $[\epsilon] \in \widetilde{A}^+ = W(\widetilde{E}^+)$
- $\log[\epsilon] \in A^+_{\max} \subset B^+_{\max} \subset B_{\max}$

Now let's give the Lubin-Tate interpretation of each of these items. In the first case, epsilon corresponds to a generator of the Lubin-Tate module because $\epsilon - 1$ is a generator of $\mathbb{Z}_p(1)$ over $\mathbb{Z}_p$.

In the second, $\widehat{p}([\epsilon]-1) = \phi([\epsilon]-1)$ where ϕ is the Frobenius morphism.

Also

$$\log([\epsilon]) = \log_{\mathbb{G}_m}([\epsilon]-1), \qquad \log_{\mathbb{G}_m}(X) : \mathbb{G}_m \xrightarrow{\sim} \mathbb{G}_a/\mathbb{Q}$$

13.4. **Next case.** We now assume f is a Lubin-Tate polynomial, e.g., $f(x) = \pi X = X^q$.

Proposition 13.3 (Colmez). *For any $x \in \widetilde{E}^+$ there is a unique $\{x\} \in \widetilde{A}^+_K$ such that*

$$\begin{aligned} \{x\} &\equiv x \mod \pi \\ f(\{x\}) &= \phi_K(\{x\}) \end{aligned}$$

For example, if $K = \mathbb{Q}_p$, $\pi = p$, $f(X) = (1+X)^p - 1$, then

$$\{x\} = [x+1] - 1$$

To check, just remember it's very easy to apply Frobenius to Teichmüller representatives by just taking lifts of p-th powers.

Proof. Idea: We want a fixed point of the map $y \mapsto \phi_K^{-1}(f(y)) =: S(y)$. Define $M := (\text{mod}\pi)^{-1}(x)$ to be the set of all liftings of x. We'd like S to act on M. But if $\widetilde{x} \in M$, then $M = \widetilde{x} + \pi\widetilde{A}^+_K$, so

$$f(\widetilde{x} + \pi y) = f(\widetilde{x}) = \widetilde{x}^q \mod \pi$$

Applying the Frobenius morphism gives

$$\phi_K^{-1} f(\widetilde{x} + \pi y) \equiv \phi_K^{-1}(f(\widetilde{x})) \equiv \phi_K^{-1}(\widetilde{x}^q) \equiv \widetilde{x} \mod \pi$$

So S acts on M.

If $a, b \in \widetilde{A}^+_K$ and $a \equiv b \mod \pi^n$, then $a^q \equiv b^q \mod \pi^{n+1}$. Since $f(X) \equiv x^q \mod \pi$ we have $f(a) \equiv f(b) \mod \pi^{r+1}$. Thus S is a p-adic contraction of the complete space M and the proof is finished.

Note that $x \mapsto \{x\}$ commutes with G_K and ϕ_K by unicity of $\{x\}$. Note also that if $\nu_{\widetilde{E}^+}(x) > 0$, then the valuation introduced in Joseph's talk satisfies

$$V(\{x\}) > 0.$$

and

$$V(\sum_i [x_i]\pi^i := \inf_i (i + \nu_{\widetilde{E}^+}(x_i))$$

$\square$

Lemma 13.4. *Let F be a formal group law over $\mathcal{O}_K$ and $\log_F : F \xrightarrow{\sim} \mathbb{G}_a$ over K where $\log_F(x) = \sum_{i\geq 0} a_i x^i$, then*

$$\nu_p(a_i) \geq -\nu_p(i)$$

Proof.

$$d\log_F = \omega_F = \sum_{i\geq 0} b_i x^i dx, \qquad b_i \in \mathcal{O}_K$$

where $b_i = i \cdot a_{i+1}$. Done. $\square$

We would like to have an interpretation of $\widetilde{E}^+$ in terms of π and not just p.

Lemma 13.5.

$$\widetilde{E}^+ \cong \varprojlim \mathcal{O}_{\mathbb{C}_p}/\pi$$

Proof. Easy exercise. $\square$

13.5. **The case of a general K.** We construct t_π. Fix a Lubin-Tate polynomial f. Take a generator (u_i) of $T(F)$ where $u_i \in \mathfrak{m}_{K^{\mathrm{alg}}} \subset \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}} \subset \mathbb{C}_p$ such that $f(u_{i+1}) = u_i, u_i \neq 0, f(u_1) = 0$.

Then $(0, u_1(\text{mod } \pi), u_2(\text{mod } \pi), \ldots) \in \widetilde{E}^+$ by the second lemma. Then we take $\{u\} \in \widetilde{A}^+$. Then $\log_F(\{u\})$ converges in $\widetilde{B}^+_{\max,K}$ by the first lemma.

Note that if $\nu_{\widetilde{E}^+}(x) > 0$, then for all $a \in \mathcal{O}_K$,

$$\{\widehat{a}(x)\} = \widehat{a}(\{x\}) \in \widetilde{B}^+_{\max,K}$$

by the unicity of $\{x\}$ and $\widehat{\pi} = f$ commutes with $\widehat{a}$.

Proposition 13.6. *For all $b \in G_K$,*

$$\sigma(t_\pi) = \chi_\pi(\sigma) \cdot t_\pi, \qquad \chi_\pi : G_k \to \mathcal{O}_K^*$$

and $\phi_K(t_\pi) = \pi \cdot t_\pi$.

Proof of the first equality. But first a lemma:

Lemma 13.7. *If F is a Lubin-Tate group law, then*

$$\log_F(\widehat{a}(x)) = a \cdot \log_F(x)$$

The proof is left as an exercise.

Now, for all $\sigma \in G_K$, we have

$$\begin{aligned}
\sigma(t_\pi) = \sigma(\log_F(\{u\})) &= \log_F(\sigma(\{x\})) = \log_F(\{\sigma u\}) \\
&= \log_f(\{\widehat{\chi_\pi(\sigma)}(u)\}) = \log_F(\widehat{\chi_\pi(\sigma)}(\{u\})) \\
&= \chi_\pi(\sigma) \cdot \log_F(\{x\})
\end{aligned}$$

Done. $\square$

A harder proposition to show is that

- $t_\pi \in \mathrm{Fil}^1 \setminus \mathrm{Fil}^2$
- $\phi^i(t_\pi) \in \mathrm{Fil}^0 \setminus \mathrm{Fil}^2$ for $0 < i \leq f - 1$.

13.6. $D_{\mathbf{cris}}(K(1))$.

Theorem 13.8. *$K(1)$ is crystalline and $D_{cris}(K(1))$ is:*

$$D := \oplus_{i=0}^{f-1} K \cdot \phi^i(v)$$

for v a formal symbol. It has dimension d over K_0. ϕ acts K-linearly. It has a K_0-linear structure by $a \mapsto (a, \phi(a), \ldots, \phi^{(f-1)}(a)) \in K \oplus \cdots \oplus K$. It has a filtration

$$Fil^{-1} := D_K, \quad Fil^1 := 0, \quad Fil^0 := \ker(D \otimes_{K_0} K \twoheadrightarrow K \cdot v \otimes_{K_0} K \to K)$$

Proof hint. Let e be a generator of $K(1)$ over K. Then

$$K(1) \otimes_{\mathbb{Q}_p} B_{\max} = K(1) \otimes_{K_0} (K_0 \otimes_{\mathbb{Q}_p} B_{\max}) = K(1) \otimes_{K_0} (\oplus_{i=0}^{f-1} B_{\max}) \supset K(1) \otimes_{K_0} B_{\max} = B_{\max,K} \cdot e,$$

which contains $t_\pi^{-1} \cdot e$. □

13.7. **Transcendence degree.** The setting is as follows.

- Let V be a p-adic G_K-representation of dimension d.
- Let $B/\mathbb{Q}_p$ be a period ring (e.g., $B = B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{st}}, B_{\max}$).
- Let V be B-admissible.
- Take an e-basis of V over $\mathbb{Q}_p$ and v-basis of $D := D_B(V)/B^{G_K}$
- Let $P \in Mat_{d\times d}(B)$ such that $e = pv$ via $V \otimes_{\mathbb{Q}_p} B \cong D \otimes_{B^{G_K}} B$.
- $P = (p_{ij}), \quad p_{ij} \in B$.
- Let $L := \mathbb{Q}_p^{\mathrm{alg}}(p_{ij})$ for simplicity, suppose $\mathbb{Q}_p^{\mathrm{alg}} \subset B$, e.g., for B_{dR}.

Finally, assume

$$G = \overline{\rho(G_K)}^{\mathrm{Zar}} \subset GL(V) \cong GL_d(\mathbb{Q}_p^{\mathrm{alg}})$$

Theorem 13.9 (Gronthendieck conjecture in the p-adic case)**.**

$$tr\ deg\ (L/\mathbb{Q}_p^{alg}) = \dim G$$

The application to our case is

$$\rho = \chi_\pi : G_K \twoheadrightarrow \mathcal{O}_K^* \subset GL_d(\mathbb{Q}_p)$$

Proposition 13.10.

$$\overline{\mathcal{O}_K^*}^{Zar} = R_{K/\mathbb{Q}_p}(\mathbb{G}_m)$$

By taking p-adic tangent spaces (Lie algebras), and noticing relation of Zariski closure with algebraic closue, we get $\supset$. Evidently $\subset$, so we're done.

Corollary 13.11.

$$tr\ deg(\text{Lubin-Tate periods}) = d$$

Proof of the Grothendieck conjecture in the p-adic case. What makes this easier than the classical case, is that, in the p-adic case, we have by definition the action of the Galois group on the period.

Consider the algebraic variety $Mat_{d\times d}(\mathbb{Q}_p^{\mathrm{alg}}) \cong \mathbb{A}^{d^2}$. Then $GL_d(\mathbb{Q}_p^{\mathrm{alg}})$ acts on $\mathbb{A}^{d^2}$. We also have

$$\mathrm{Spec}\ L \to \mathbb{A}^{d^2}_{\mathbb{Q}_p^{\mathrm{alg}}}$$

given by the (p_{ij}) denote its closure by X.

We assume the strong statement thattr deg $L = \dim(X)$.

Consider the Tannakian formalism. For us, it is a black box that implies $X \subset \{G-\text{torsor}\} \subset \mathbb{A}^{d^2}$.

Then X is G_0-invariant, where G_0 is the connected component of the identity in G.

Suppose that $f(p_{ij}) = 0$ for $f \in F[T_{ij}]$ and F/K is a finite extension. Let $\sigma \in G_F$. How does σ act on p? We know $e = p \cdot v$ so

$$\rho(\sigma)p \cdot v = \rho(\sigma)e = \sigma(e) = \sigma(p) \cdot \sigma(v) = \sigma(p) \cdot v$$

Then $\sigma(f(p_{ij})) = 0$ but also $\sigma(f(p_{ij})) = f(\rho(\sigma) \cdot p) = f^{\rho(G)}(p_{ij})$.

So $f^{\overline{G_F}^{\mathrm{Zar}}}$ vanishes on X.

Since $G_F \subset G_K$ is of finite index, $\overline{G_F}^{\mathrm{Zar}}$ and $G = \overline{G_K}^{\mathrm{Zar}}$ have the same connected component of the identity. So f^{G_0} is zero on X and thus X is G_0-invariant. Done! □

14. Why are de Rham representations potentially semi-stable?

An informal lecture of Laurent Berger on Thursday, the 22nd of July, 2010.

Let $K/\mathbb{Q}_p$ be a finite extension. Remember we denote p-adic completion by the hat $\widehat{\cdot}$. We've already seen that

$$\begin{array}{rcl} \{p\text{-adic representations of } G_K\} & & \\ \cup & & \\ \{\text{de Rham}\} & = & \{\text{potentially de Rham}\} \\ \cup & & \cup(*) \\ \{\text{semi-stable}\} & \subset & \{\text{ potentially semi-stable}\} \\ \cup & & \cup \\ \{\text{crystalline}\} & \subset & \{\text{potentially crystalline}\} \end{array}$$

Conjecture 14.1 (Fontaine)**.** *The inclusion* $(*)$ *is an equality.*

This is now a theorem of André, Kedlaya, and Mebkhoot (three separate proofs).

The ingredients of the proof are (ϕ, Γ)-modules. For simplicity, we'll consider representations of $G_{\mathbb{Q}_p}$.

Let $\rho \in \mathbb{R}_{<1}$. Then

$$\mathcal{E}_K^{\dagger,\rho} := \left\{ f(x) = \sum_{i \in \mathbb{Z}} a_i X^i \mid a_i \in K, f(x) \text{ converges and is bounded on } \{x \in \mathbb{C}_p \mid \rho < |X|_p < 1\} \right\}.$$

Then

$$\cup_{\rho<1} \mathcal{E}_K^{\dagger,\rho} = \mathcal{E}_K^\dagger \supset \mathcal{O}_{\mathcal{E}_K}^\dagger = \{f(x) \mid |a_i| \leq 1\}$$

Note that $\mathcal{E}_K^\dagger$ is a field.

Now let ϕ be the Frobenius morphism on $\mathcal{E}_K^\dagger$, and

$$(\phi f)(x) = f((1+X)^p - 1)$$

Then

$$\Gamma = \Gamma_K = G_{K^{\mathrm{alg}}}/G_K = \mathrm{Gal}(K_\infty/K) \hookrightarrow \mathbb{Z}_p^*$$

where the last map is a character χ. Then define an action of $\gamma \in \Gamma_K$ on $\mathcal{E}_K^\dagger$ by

$$(\gamma f) := f((1+X)^{\chi(\gamma)} - 1)$$

Definition 14.2. A (ϕ, Γ)-module over $\mathcal{E}_K^\dagger$ is a finite dimensional vector space D over $\mathcal{E}_K^\dagger$ such that

- $\phi : D \to D$ a σ-semilinear map with $Mat(\phi) \in GL_d(\mathcal{E}_K^\dagger)$.
- Γ acts on D, is σ-semilinear, and commutes with ϕ.

Recall that $\sigma := W(x \mapsto x^p)$ denotes the absolute Frobenius morphism.

Then there is a functor

$$\{(\phi, \Gamma)\text{-modules over } \mathcal{E}_K^\dagger\} \to \{p\text{-adic representations of } G_K\}.$$

There are inclusions $\widetilde{E}^+ \hookrightarrow \widetilde{A}^+ \hookrightarrow \widetilde{B}^+ \hookrightarrow \cdots$. We also have $\widetilde{A} = W(\widetilde{E})$ and $\widetilde{B} = \widetilde{A}[1/p]$. Then $\widetilde{B}$ is a field with an aciton of G_K and a Frobenius morphism $\phi = W(y \mapsto y^p)$.

As in previous lectures, let $\pi = [\epsilon] - 1 \in \widetilde{B}$. Then $\mathcal{E}_K^\dagger \subset \widehat{\mathbb{Z}_p[[\pi]]}\left[\frac{1}{\pi}\right] = B_{\mathbb{Q}_p} \subset \widetilde{B}$. The inclusion of $\mathcal{E}_K^\dagger$ is given by $x \mapsto \pi$ and is compatible with both the G_K action and ϕ.

Also,

$$\widetilde{B}^{\phi=1} = \mathbb{Q}_p = \left(W(\widetilde{E})\left[\frac{1}{p}\right]\right)^{\phi=1} = W(\mathbb{F}_p)\left[\frac{1}{p}\right]$$

Then

$$D \mapsto V(D) = (\widetilde{B} \otimes_{\mathcal{E}_K^\dagger} D)^{\phi=1}$$

is a $\mathbb{Q}_p$-vector space with an action of G_K given by $g(b \otimes d) = g(b) \otimes \overline{g}(d)$ where $\overline{g} \in \Gamma_K$.

Definition 14.3. A (ϕ, Γ)-module D over $\mathcal{E}_K^\dagger$ is étale if there exists a basis in which $Mat(\phi) \in GL_d(\mathcal{O}_{\mathcal{E}_K}^\dagger)$.

Theorem 14.4 (Easy)**.** *If D is étale, then $V(D)$ is a $\mathbb{Q}_p$-vector space of dimension* $\dim_{\mathcal{E}_K^\dagger} D$.

A hard theorem is the following.

Theorem 14.5 (Cherboniev, Colmez)**.** *The resulting functor,*

$$\begin{array}{rcl} \{\textit{étale}\,(\phi, \Gamma)\textit{-module}\} & \to & \{p\textit{-adic representations}\} \\ D & \mapsto & V(D), \end{array}$$

is an equivalence of categories.

Then $V \mapsto (\widetilde{B} \otimes V)^{H_K}$ is a $\widetilde{B}_K$-vector space of dimension $d = \dim V$. But we have the inclusions

$$\widetilde{B}_K = \widehat{\cup \phi^{-n}(B_K)} \supset B_K \to \mathcal{E}_K^\dagger$$

The proof by Colmez uses

$$H^1(G_K, GL_d(\mathbb{Q}_p)) \to H^1(G_K, GL_d(\widetilde{B}_K^\dagger)).$$

This is technical. For more information, see the notes of Berger.

The conclusion is $D(V) \subset \widetilde{B} \otimes_{\mathbb{Q}_p} V \subset \widetilde{B}^\dagger \otimes_{\mathbb{Q}_p} V$.

$$\widetilde{B}^\dagger = \left\{ x = \sum_{n \gg -\infty} p^n [x_n] \in \widetilde{B} \mid x_n \in \widetilde{E}, \exists \sigma = \sigma(x), \nu(x_n) + \sigma \cdot n \to +\infty \right\}$$

Now we'll define a few other rings. They are almost the same.

Definition 14.6. Let $\rho \in \mathbb{R}_{<1}$. Then

$$\mathcal{R}_K^{\dagger,\rho} = \left\{ f(x) = \sum_i a_i X^i \mid f(x) \text{ converges on } \{x \in \mathbb{C}_p \mid \rho < |x|_p < 1\} \right\}$$

Note we've dropped the boundedness condition in the definition of $\mathcal{E}_K^{\dagger,\rho}$. Also define the "Robba ring" as

$$\mathcal{R}_K^\dagger := \cup_{\rho<1} \mathcal{R}_K^{\dagger,\rho} \quad (= \mathcal{R}_K).$$

We will usually denote it by $\mathcal{R}_K$, dropping the dagger.

Let $t = \log(1+X) \in \mathcal{R}_K^{\dagger,\rho}$. Then the definitions of ϕ, Γ, etc. for $\mathcal{E}_K^\dagger$ extend to definitions for $\mathcal{R}_K$.

If V is a p-adic representation, then $D(V)$ is a (ϕ,Γ)-module over $\mathcal{E}_K^\dagger$, and

$$D_{\text{rig}}(V) := \mathcal{R}_K \otimes_{\mathcal{E}_K^\dagger} V$$

is a (ϕ,Γ)-module over $\mathcal{R}_K$.

Theorem 14.7. *If V is a representation of G_K, then*

$$D_{cris}(V) = \left(D_{rig}(V) \left[\frac{1}{t} \right] \right)^{\Gamma_K}$$

In particular, these are vector spaces of the same dimension, and in fact, are isomorphic as ϕ-modules. We can define a filtration on the right-hand side using the order of vanishing at $\zeta_{p^n} - 1$ for $n \gg 0$.

Theorem 14.8. *Consider*

$$\phi(\log X) = p \log X + \log \frac{X}{X^p} \in D_{rig}(V) \left[\frac{1}{t}, \log X \right]$$

and the action of γ by $\gamma(\log X) = \log X + \log \frac{\gamma(X)}{X}$. Because we just added a variable to the ring, there is a new operator which is derivation with respect to X. The point is, there is a relation between the X and the N. Then

$$D_{st}(V) = \left(D_{rig}(V) \left[\frac{1}{t}, \log X \right] \right)^{\Gamma_K}$$

these are isomorphic as ϕ-modules.

Proof. Recall that in one of the talks, we had

$$\widetilde{B}^+_{\text{rig}} = \cap_{n \geq 1} \phi^n (B^+_{\text{max}}).$$

An easy exercise shows that if V is a p-adic representation, then $D_{\text{rig}}(V) = \left(\widetilde{B}^+_{\text{rig}} \left[\frac{1}{t} \right] \otimes V \right)^{G_K}$.

Then the main idea of the proof is that one can construct a $\widetilde{B}^{\dagger}_{\text{rig}}$ which has contained most of the rings we've seen: $\widetilde{B}^{+}_{\text{rig}}$ and $\widetilde{B}^{\dagger}$ and $\mathcal{R}_K$. Then $D_{\text{cris}}(V) = (\widetilde{B}^{+}_{\text{rig}}\left[\frac{1}{t}\right] \otimes V)^{G_K} \supset \widetilde{B}^{\dagger}_{\text{rig}}\left[\frac{1}{t}\right] \otimes V$.

Then there is a tower of inclusions

$$\left(D_{\text{rig}}(V)\left[\frac{1}{t}\right]\right)^{\Gamma_K} \subset \mathcal{R}_K \otimes (\widetilde{B}^{\dagger} \otimes V)\left[\frac{1}{t}\right] \subset \widetilde{B}^{\dagger}_{\text{rig}}\left[\frac{1}{t}\right] \otimes V.$$

Define

$$\widetilde{B}^{\dagger,\sigma} := \{\sum_n p^n[x_n] \mid \nu_E(x_n + +\sigma \cdot n \to +\infty\}$$

and $V(x,\sigma) := \min_{n\in\mathbb{Z}}(\nu_E(x_n) + n \cdot \sigma)$ and $V(x,\tau) := \min_{n\in\mathbb{Z}}(\nu_E(x_n) + n \cdot \tau)$ if $\tau > \sigma$. Also, $V(x,[\sigma,\tau]) = \min[V(x,\sigma), V(x,\tau)]$.

Also define

$$\widetilde{B}^{\dagger,\sigma}_{\text{rig}} := \text{ completion of } \widetilde{B}^{\dagger,\sigma} \text{ for all } V(\cdot,[\sigma,\tau]), \qquad \tau \geq \sigma$$

and $\widetilde{B}^{\dagger}_{\text{rig}} := \cup_{\sigma>0}\widetilde{B}^{\dagger,\rho}_{\text{rig}}$.

If $\sigma = 1$, then we can examine $\widetilde{B}^{\dagger,1}$ and its completion $\widetilde{B}^{\dagger,1}_{\text{rig}}$. Then each term of the series in $\widetilde{B}^{\dagger,\sigma}$ go to zero, so $\widetilde{B}^{\dagger,1}_{\text{rig}} \hookrightarrow B^{+}_{\text{dR}}$.

By this we can tell from $D_{\text{rig}}(V)$ whether V is de Rham.

$$D_{\text{rig}}(V) = \mathcal{R}_K \otimes_{\mathcal{R}_K^{\dagger,\rho}} D^{\rho}_{\text{rig}}(V), \qquad \rho < 1$$

Then exists a basis in which $Mat(\text{everything}) \in M_{d\times d}(\mathcal{R}_K^{\dagger,\rho})$. Then the morphism

$$\begin{array}{rcl} D^{\rho}_{\text{rig}}(V) & \to & B^{+}_{\text{dR}} \otimes V \\ y & \mapsto & \phi^{-n}(y) \end{array}$$

is well-defined for $n > n(\rho)$, where $n(\rho)$ is a lower bound depending on ρ.

Assume $n > n(\rho)$. We can check that the condition $\rho < |\zeta_p - 1|_p$ is the same as the condition that $\sum_n \exp(\frac{t}{p^n} - 1$ converges in the local ring $K_n[[t]]$, where $K_n = K(\zeta_{p^n})$ and $t = \log(1+X)$. This requires the map

$$\eta_n : \mathcal{R}_K^{\dagger,\rho} \to K_n[[t]], \quad f(x) \mapsto f\left(\sum_n \exp(t/p^n) - 1\right)$$

The morphisms η_n commute with G_K.

Let $\rho < 1$ as usual. Examine

$$\begin{array}{rcl} V & \mapsto & D^{\rho}_{\text{rig}}(V) \\ & \mapsto & K_n[[t]] \otimes_{\mathcal{R}_K^{\dagger,\rho}} D^{\rho}_{\text{rig}}(V) \\ & \mapsto & \text{a } K_n[[t]]\text{-module and a connection } \nabla m = \lim\limits_{\gamma\to 1} \dfrac{(\gamma-1)m}{\log_p \chi(\gamma)} \end{array}$$

The second line is a free $K_n[[t]]$-module of rank d with an aciton of Γ_K. The connection in the third line satisfies $\nabla(f(t)\cdot m) = t\frac{df}{dt}\cdot m + f(t)\nabla m$.

To finish things off, we'll require the following theorem.

Theorem 14.9 (Fontaine)**.** *V is de Rham iff $(K_n[[t]] \otimes D^\rho_{rig}(V), \nabla) =: M_n$ is trivial. In this case, $M_n[1/f]^{\nabla=0} = K_n \otimes_K D_{dR}(V)$.*

We've now found a local condition which tells us whether V is de Rham. Finally, we want to reduce this case to statements about p-adic differential equation. We will find a local criterion in terms on equations, and then glue them together to get a global criteria.

So we consider $D^\rho_{\text{rig}}(V)$ a $\mathcal{R}^{\dagger,\rho}_K$-module with a Γ_K-action. We may, as before, define an action of the Lie algebra by

$$\nabla_m = \lim_{\gamma\to 0} \frac{(\gamma-1)m}{\log_p \chi(\sigma)} \in D^\rho_{\text{rig}}(V)$$

and

$$\nabla(f(x)\cdot m) = (1+X)\log(1+X)\frac{df}{dX}m + f(x)\nabla_m$$

But $\log(1+X)$ has infinitely many zeros, so it is not invertible. This is not good, because it shows we can't define a p-adic differential equation for each representation. But we use Fontaine's theorem and find that $1/t$ preserves some submodule of $(K_n[[t]] \otimes D^\rho_{\text{rig}}(V), \nabla)$.

If V is de Rham, define

$$N^\rho_{\text{dR}} = \{y \in D^\rho_{\text{rig}}(V)\left[\frac{1}{t}\right] \mid n > n(\rho), \eta_n(y) \in K_n[[t]] \otimes_K D_{\text{dR}}(V)\}$$

We're almost done. We just need to finish reducing to p-adic differential equations. Consider the following results toward that end.

Theorem 14.10. *If V is a de Rham representation, then*

- *N^ρ_{dR} is a free $\mathcal{R}^{\dagger,\rho}_K$-module of rank d*
- *$N^\rho_{dR}\left[\frac{1}{t}\right] = D^\rho_{rig}(V)\left[\frac{1}{t}\right]$*
- *$\nabla(N^\rho_{dR}(V)) \subset t \cdot N^\rho_{dR}$*

Corollary 14.11. *If we set $\partial = \frac{1}{t}\nabla$ and $N_{dR}(V) = \mathcal{R}_K \otimes_{cR^{\dagger,\rho}_K} N^\rho_{dR}(V)$, then $(N_{dR}(V), \partial)$ is a p-adic differential equation over $\mathcal{R}_K$ with*

$$\partial(f(x)m) = (1+X)\frac{df}{dX}m + f(X)\partial m$$

Theorem 14.12 (André, Kedlaya, Mebkhout)**.** *This is a former conjecture of Crew and Tsuzuki. If (M, ∂) is a p-adic differential equation over cR_K, then there exists a finite extension L/K such that*

$$\mathcal{R}_L[\log X] \otimes_{\mathcal{R}_K} M)^{\partial=0}$$

is a $\mathcal{R}^{\partial=0}_L$-vector space of dimension of the same rank as M if

- *M has a Frobenius phi which is semilinear and $\partial\phi = p\phi\partial$.*
- *or some other conditions*

Now we've finished the proof. Let's close with a summary. Given V a de Rham representation, we associate a p-adic differential equation $(N_{\text{dR}}, \partial)$. Then there exists a finite extension L/K such that $(\mathcal{R}_L[\log X] \otimes_{\mathcal{R}_K} N_{\text{dR}})^{\partial=0}$ is of dimension d. Thus $\left(D_{\text{rig}}(V_{|L_n})\left[\frac{1}{t}, \log X\right]\right)^{\Gamma_{L_n}}$ is of dimension d. Thus we conclude that $V_{|L_n}$ is semi-stable. □

15. From classical to p-adic Hodge theory

Brent Doran on the 23rd of July, 2010.

Advertisement: Hodge theory course in the fall. Topics covered will may be heavily swayed by participants.

In this course, we haven't discussed classical Hodge theory, i.e., Hodge theory over $\mathbb{C}$. We've gone straight to p-adic theory. In the classical case, we have a meromorphic form ω which we integrate over cycles $u \in H^1$.

An important example are abelian varieties, such as the Jacobians of curves and the intermediate Jacobians of cubic 3-folds. The techniques include

- Theta functions
- Riemann bilinear relations

Our goal today is to give a construction of p-adic periods for abelian varieties that's as close as possible to the pattern of the classical case.

Issues to overcome are

(1) What is the p-adic analogue of a 1-cycle on an abelian variety?
(2) What if a meromorphic form ω has pole on the 1-cycle u? (Topologically, we can slide u to a homologous 1-cycle.)
(3) How do we define the period pairing given by integration
$$\int : H_1(X(\mathbb{C}), \mathbb{Z}) \times H^1_{\mathrm{dR}}(X) \to \mathbb{C},$$
and, in particular, what field should play the role of $\mathbb{C}$ and contain the periods?

15.1. Issue one.

Now restrict to the case that X is an abelian variety over K such that $K^{\mathrm{alg}} \subset \mathbb{C}$ or $\mathbb{C}_p$. Fix this algebraic closure. Let $u \in H_1(X(\mathbb{C}), \mathbb{Z})$ be a 1-cycle in the singular homology of the topological space $X(\mathbb{C})$. There is a projection map $\mathrm{pr} : \mathbb{C}^d \to X(\mathbb{C})$. Lift u through this projection and let $i(u)$ be one of its end-points, which we'll call a basepoint. Let $T_p(X)$ be the p-adic Tate module,
$$T_p(X) = \varprojlim X[p^n],$$
where $X[p^n] = \{x \in X \mid x^{p^n} = 1\}$ is the p-torsion. Then there is a map
$$\begin{array}{rcl} H_1(X(\mathbb{C}), \mathbb{Z}) & \to & T_p(X) \\ u & \mapsto & (0, \ldots, \mathrm{pr}(p^n i(u)), \ldots). \end{array}$$
This induces an isomorphism $H_1(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_p \xrightarrow{\sim} T_p(X)$.

15.2. Issues two and three.

To overcome issues (ii) and (iii) properly, we'll need analogs of complex results. To begin, let
$$H^1_{\mathrm{dR}}(X) = \frac{\{\text{1-forms of the second kind}\}}{\{\text{exact 1-forms}\}}$$
where the exact one forms are exactly the differentials of rational functions. There is a filtration
$$\begin{array}{ccccc} H^1_{\mathrm{dR}}(X) & \supset & H^0(X, \Omega^1_X) & \supset & 0 \\ \mathrm{Fil}^0 & \supset & \mathrm{Fil}^1 & \supset & \mathrm{Fil}^2 \end{array}$$

Given ω a boundary of a 2-form on X/K, then let f_ω be a meromorphic function on X such that

$$df_\omega = \mathrm{pr}^*\omega \qquad \text{on } \mathbb{C}^d$$

where we can pull back because $\mathbb{C}^d$ is contractible. If $u \in H_1(X(\mathbb{C}), \mathbb{Z})$ and $a \in \mathbb{C}^d$ not a pole of f_ω, then

$$f_\omega(i(u) + a) - f_\omega(a)$$

depends only on u and the class of ω in $H^1_{\mathrm{dR}}(X)$. It is the data of the integral $\int_u \omega$. We can use the group law to slide the 1-cycle ω off the pole. This gives a pairing

$$\begin{array}{rcl} H_1(X(\mathbb{C}), \mathbb{Z}) \times H^1_{\mathrm{dR}}(X) & \to & \mathbb{C} \\ (u, \omega) & \mapsto & f_\omega(i(u) + a) - f_\omega(a) \end{array}$$

which is

- bilinear and
- non-degenerate upon extension of scalars to $\mathbb{C}$.

Proposition 15.1. *To each f_ω, we can associate a purely algebraic function, F_ω which is uniquely defined up to constant.*

Proof. We'll show it later. for now, we'll explain the idea. □

Consider $f_\omega(z_0 + z_1 + z_2) - f_\omega(z_0 + z_1) - f_\omega(z_0 + z_2) + f_\omega(z_0)$. This is $(\mathbb{C}^*)^3$ periodic with Λ^3 its lattice. It induces a rational function F^3_ω on X^3/K. It is algebraically characterized by the two facts that

(1) $F^3_\omega(X_0, 0, X_2) = F^3_\omega(X_0, X_1, 0) = 0$
(2) The differential is

$$m^*_{\{0,1,2\}}\omega - m^*_{\{0,1\}}\omega - m^*_{\{0,2\}}\omega + m^*_{\{0\}}\omega$$

where $I \subset \{0, 1, 2\}$. For each I, we get a morphism $m_I : X^3 \to X$ given by $(x_1, x_2, x_3) \mapsto \oplus_{i \in I} x_i$, where the sum $\oplus$ is done in the group law of the abelian variety X.

In the p-adic case, we should replace the period pairing with the map

$$\begin{array}{rcl} T_p(X) \times H^1_{\mathrm{dR}}(X) & \to & B^+_{\mathrm{dR}} \quad (\text{or } B_{\mathrm{dR}}) \\ (u, \omega) & \mapsto & \lim_{n\to\infty} p^n(F_\omega(a_n) - F_\omega(a_n \oplus \widehat{u_n}) \end{array}$$

where we assume that neither of the arguments of F_ω are poles of ω, and the sum $\oplus$ is in the group law on X.

So our next task is to define these, show converge and show independence from the choices made in the definitions.

15.3. **Various rings.** We recall here various rings we've seen. For starters, look at B^+_{dR}. We have rings of integers $\mathcal{O} \subset \mathbb{Q}_p^{\mathrm{alg}}$ and $\mathcal{O}_{\mathbb{C}_p} \subset \mathbb{C}_p$. There is a valuation $\widehat{\nu}$ on $\mathbb{C}_p$ such that $\widehat{nu}(p) = 1$, and there is a norm $|x|_p = p^{-\widehat{\nu}(x)}$ as usual. Let

$$\widetilde{E}^+ = \{x = (x^{(n)}) \in \mathcal{O}_{\mathbb{C}_p} \mid (x^{(n+1)})^p = (x^{(n)})\}.$$

Consider the Witt vectors $W(\widetilde{E}^+)$ and the morphism $\theta : W(\widetilde{E}^+) \to \mathcal{O}_{\mathbb{C}_p}$ which can be extended to $\widetilde{B}^+ := W(\widetilde{E}^+)[p^{-1}]$, and finally to B_{dR}, the field of fractions of B^+_{dR}.

Definition 15.2. Let $K/\mathbb{Q}_p$ be a finite extension and $\mathcal{O}_K$ the ring of integers. Define

$$\widetilde{A}_K^+ := A_{\mathrm{inf},K} := \text{the smallest ring of } B_{\mathrm{dR}}^+ \text{ generated by } W(\widetilde{E}^+) \text{ and } \mathcal{O}_K$$

Then $\ker\theta \cap \widetilde{A}_K^+$ is a principal idela in $\widetilde{A}_K^+$.

Let π be a uniformizer for $\mathcal{O}_K$ and e a generator of $\ker\theta \cap \widetilde{A}_K^+$.

Definition 15.3. If $b \in \mathbb{N}$, let

$$\widetilde{A}_K^{\dagger,b} = A_{\mathrm{inf},K}^b := \widetilde{A}_K^+[[\pi^{-b}\rho]]$$

be the closed subring of B_{dR}^+. It doesn't depend on the choices of π and ρ.

15.4. **Past motivation.** Much of the basic structure follows from the group law. The setting is a finite extension $K/\mathbb{Q}_p$ with a maximal unramified subextension K_0. We'll call the Frobenius morphism $\sigma : K \to K$. Let G be a commutative formal group law on $\mathcal{O}_K$ of dimension d of height h. We'll denote sum in the formal gropu law as $\oplus$, and $[n] \in \mathrm{End}(G)$. Let $\mathcal{O}_K[[X_1, \ldots, X_d]]$ be the affine algebra of G and define a closed differential form

$$\omega = \sum_{i=1}^{d} \alpha_i(X_1, \ldots, X_d)dx, \qquad \alpha_i(X) \in K[[X_1, \ldots, X_d]].$$

Definition 15.4. Define an algebraic function

$$F_\omega := \text{ the unique element of } K[[X_1, \ldots, X_d]] \text{ such that } dF_\omega = \omega \text{ and } F_\omega(0) = 0$$

Let F_ω^2 be an element of $K[[X_1, \ldots, X_d, Y_1, \ldots, Y_d]]$ given by

$$F_\omega^2 := F_\omega(X \oplus Y) - F_\omega(X) - F_\omega(Y).$$

Now for some classes of closed differential forms

Definition 15.5. Let ω be a closed differential form. Then we say ω is exact if there exists an $r \in \mathbb{N}$ such that $\pi^r F_\omega \in \mathcal{O}_K[[X_1, \ldots, X_d]]$. We say that ω is invariant if $F_\omega^2 = 0$. We say that it is of the second kind if there exists an $r \in \mathbb{N}$ such that $\pi^r F_\omega^2 \in \mathcal{O}_K[[X_1, \ldots, X_d]]$

Let Ω be the K-vector space of invariant differential forms. It is of dimension d.

Let

$$H^1_{\mathrm{dR}}(G) := \frac{\text{1-forms of the second kind}}{\text{exact 1-forms}}$$

It is filtered as $H^1_{\mathrm{dR}}(G) \supset \Omega_G \supset 0$. But $H^1_{\mathrm{dR}}(G)$ also has a sub-K_0-vector space $D(G) := \{$ differential forms with coefficients in K_0 $\}$. The first cohomology decomposes as

$$H^1_{\mathrm{dR}}(G) \cong K \otimes_{K_0} D(G).$$

It is equipped with a σ-semi-linear action given by

$$\phi(\omega) = \omega^\sigma((X_1)^p, \ldots, (X_d)^p)$$

Then $T_p(\omega$ (how is it defined?) is a $\mathbb{Z}_p$-module of rank h with a $\mathrm{Gal}(K^{\mathrm{alg}}/K)$-action.

Proposition 15.6. *Let ω be a 1-form of the second kind. Let $u = (0, \ldots, u_n, \ldots) \in T_p(G)$. Then there is a lifting $\widehat{u_n} \in (\widetilde{A}_K^+)^d$ such that $\mathcal{O}(\widehat{u_n}) = u_n$. The following are true:*

1. *The sequence $-p^n F_\omega(\widehat{u_n})$ converges in $B^+_{cris,K}$ to a limit that depends only on u and the image of u in $H^1_{dR}(G)$.*

(2) *The period map thus defined is*
- *bilinear*
- *respects filtrations, i.e.,* $\int_\omega \in Fil^1(B^+_{dR})$ *for all* $\omega \in \Omega_G$,
- *and commutes with the action of* $Gal(K^{alg}/K)$, *i.e.,* $g\int_u \omega = \int_{g(u)} \omega$.

(3) *If* $\omega \in D(G)$, *then* $\int_u \omega \in B^+_{cris}$ *and* $\phi(\int_u \omega) = \int_u \phi(\omega)$.

15.5. **Extend by analogy to abelian varieties.** Again, let $K/\mathbb{Q}_p$ be a finite extension, and X a smooth, proper algebriac variety over k of dimension d.

Definition 15.7. A map

$$F : X(B^+_{\mathrm{dR}}) \to B^+_{\mathrm{dR}}$$

is called locally analytic if for all $x \in X(B^+_{\mathrm{dR}})$ and for some (indeed, any) choice of local parameters at x, call them $z_1, \ldots, z_d$, there exists $F_x \in B^+_{\mathrm{dR}}([z_1, \ldots, z_d])$ and $r \in \mathbb{R}$ such that

(1) $F_x(z_1, \ldots, z_d)$ converges if $|\theta(z_i)|_p < r$.
(2) F_x coincides with F in a neighborhood of x.

Definition 15.8. A locally meromorphic function is a quotient of two locally analytic functions.

For example, rational functions on X are locally meromorphic.

For another example, consider a closed, rational 1-form of the second kind, ω. Then there exists a locally meromrophic function F_ω such that $dF_\omega = \omega$.

Note that F_ω is only determined up to a locally constant function.

Let's move back to the setting of abelian varieties.

Proposition 15.9. *Let* ω *be a 1-form of the second kind on* X *and* F^3_ω *be a rational function on* X^3. *Then there exists a locally meromorphic function* F_ω *on* $X(B^+_{dR})$ *which is unique up to addition by a constant such that*

(1) $dF_\omega = \omega$, *and*
(2) F_ω *and* F^3_ω *are related by the "law of the cube".*

A short discussion with the audience revealed we're encoding the fact the if you take any ample line bundle (think F_ω) on an abelian variety, then it's cube (think F^3_ω) is very ample. Note that in the classical setting F_ω would be a function on the covering space $\mathbb{C}^r$ and not on the variety X.

Back to the talk, F_ω plays the role of the integral. Let X be an abelian variety over $\mathbb{C}_p$ and $\mathcal{X}$ be a proper model of X on $\mathcal{O}_K$. Let $\mathcal{O}(\mathcal{O}_{\mathbb{C}_p})$. Let $b \gg 0$. Let ω be a 1-form of the second kind on X, and let $u = (\ldots, u_n, \ldots)$ where $u_n \in \mathcal{X}(\mathcal{O}_{\mathbb{C}_p})$. Choose lifts $\widehat{u_n} \in \mathcal{X}(\widetilde{A}^{\dagger,b}_K$ such that $\theta(\widehat{u_n}) = u_n$.

Finally, let $a_n \in U_{\omega,\widehat{u_n}}(\widetilde{A}^{\dagger,b}_K)$, which is a Zariski open subset of X avoiding poles of ω.

Proposition 15.10. *The sequence*

$$p^n(F_\omega(a_n) - F_\omega(a_n \oplus \widehat{u_n})$$

has a limit in B^+_{dR} *which only depends on* u *and the class of* $\omega \in H^1_{dR}(X)$.

Proposition 15.11. *The pairing*

$$\begin{aligned} H^1_{dR}(X) \times T_p(X) &\to B^+_{dR} \\ (\omega, u) &\mapsto \int_u \omega \end{aligned}$$

is

(1) *bilinear,*
(2) *commutes with* $Gal(K^{alg}/K)$*,*
(3) *respects filtration,*
(4) *and is non-degenerate when extending scalars to* B_{dR}*.*

Proposition 15.12. *We have p-adic theta functions and Riemann bilinear relations.*

16. Logarithm maps

Sergey Rybakov on Friday, the 23rd of July, 2010.

We define the logarithm. Let

$$1 + \mathfrak{m}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p \mid \nu(x-1) > 0\} \subset \mathbb{C}_p^*.$$

Definition 16.1. The logarithm is the map

$$\begin{aligned} 1 + \mathfrak{m}_{\mathbb{C}_p} &\to \mathbb{C}_p \\ x &\mapsto -\sum_{n \geq 0} \frac{(1-x)^n}{n} \end{aligned}$$

Lemma 16.2. *There is an exact sequence*

$$0 \to \mu_{p_\infty} \to 1 + \mathfrak{m}_{\mathbb{C}_p} \to \mathbb{C}_p \to 0$$

where $\mu_{p_\infty} = \cup_n \mu_{p^n}$*,* $\mu_{p^n} = \{x \in \mathbb{C}_p \mid x^{p^n} = 1\}$ *is the group of all p-power roots of unity in* $\mathbb{C}_p$*.*

Proof sketch. First, note that log is invertible with inverse exp on the subgroup

$$\{x \in 1 + \mathfrak{m}_{\mathbb{C}_p} \mid \nu(x-1) \geq 1\}.$$

We'd like to show the second map is surjective. Note that for all $y \in \mathbb{C}_p$, there exists $m \in \mathbb{N}$ such that $y \cdot p^m \in p\mathcal{O}_{\mathbb{C}_p}$. Define $x := (\exp(y \cdot p^m))^{1/p^m}$. Then

- $\log x = y$,
- and $x \in 1 + \mathfrak{m}_{\mathbb{C}_p}$.

So we see that the second map in the exact sequence is surjective. Now we compute the kernel.

Assume $\log x = 0$. Then there exists a $m \in \mathbb{N}$ such that

$$\nu(1 - x^{p^m}) \geq 1.$$

Since the function log is invertible, $\log x^{p^m} = 0$ iff $x^{p^m} = 1$, i.e., $x \in \mu_{p^m}$. So we've found the kernel. □

Definition 16.3. Define the subgroups

$$\begin{array}{rcl} \widetilde{E}^+ & & \\ \cup & & \\ U^+ & := & \{x \in \widetilde{E}^+ \mid \nu(x-1) > 0\} = \{x \in etp \mid x^{(0)} \in 1 + \mathfrak{m}_{\mathbb{C}_p}\} \\ \cup & & \\ U' & := & \{x \in U^+ \mid \nu(x-1) \geq 1\} \end{array}$$

Also define the logarithm,

$$\begin{array}{rcl} \log : U^+ & \to & \mathbb{C}_p \\ x & \mapsto & \log x^{(0)} \end{array}$$

Lemma 16.4. *There is a short exact sequence*

$$0 \to \mathbb{Q}_p(1) \to U^+ \to \mathbb{C}_p \to 0$$

where the first map sends 1 to $\epsilon = (1, \zeta_p, \zeta_{p^2}, \ldots)$ *and* $\zeta_p \neq 1$.

Proof. The surjectivity of the second map follows from the first lemma. The kernel is $\ker \log \cong \mathbb{Q}_p$ since

$$\begin{array}{rcl} \log x = 0 & \iff & \log x^{(0)} = 0 \\ & \iff & x^{(0)} \in \mu_{p^n}, \qquad n \gg 0 \\ & \iff & (x^{p^n})^0 = 1, \qquad x^{p^n} \in Z_p\epsilon \end{array}$$

□

Now we extend the definition of the logarithm to U^+.

Definition 16.5. Define the logarithm to be

$$\begin{array}{rcl} \log[] : U^+ & \to & B^+_{\max} \\ [x] & \mapsto & -\sum_n \frac{(1-[x])^n}{n} \end{array}$$

It converges in $A^+_{\max}$.

Proposition 16.6. *For all* $x \in U^+$, *there exists an* m *such that* $x^{p^m} \in U^1$. *Then*

$$\log[x] = \frac{1}{p^n} \log[x^{p^n}].$$

Proposition 16.7. *Let* $U := \log(U^+) \subset B^+_{max}$. *Then* $U^+ \xrightarrow{\sim} U$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & U^+ & \xrightarrow{\log} & \mathbb{C}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \cong \log} & & {\scriptstyle id}\downarrow & & \\ 0 & \longrightarrow & \mathbb{Q}_p \cdot t & \longrightarrow & U & \xrightarrow[\theta]{} & \mathbb{C}_p & \longrightarrow & 0 \end{array}$$

Proof. Note that

$$\theta([x] - 1) = x^{(0)} - 1$$

Then $\ker \log[] \subset \ker \log \cong \mathbb{Q}_p \cdot \epsilon$ where the map is $\epsilon \mapsto t$. thus $\log[] : U^+ \xrightarrow{\sim} U$. □

Theorem 16.8. *There is an exact sequence called the fundamental exact sequence,*

$$0 \to \mathbb{Q}_p \to B_{max}^{\phi=1} \to B_{dR}/B_{dR}^+ \to 0$$

Proof. First note that $\phi \log[x] = \log[x^p] = p \log[x]$, so there is an isomorphism

$$U^+ \xrightarrow{\sim} U \subset B_{\max}\phi = 1 \cap B_{\mathrm{dR}}^+.$$

Thus

$$U \cdot t^{-1} \subset B_{\max}^{\phi=1} \cap \mathrm{Fil}^{-1} B_{\mathrm{dR}}$$

Second, there is an exact sequence

$$0 \to \mathbb{Q}_p \to B_{\max}^{\phi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+.$$

The result will then follow from a proposition.

Proposition 16.9. (1) *There is a surjective morphism $B_{max}^{\phi=1} \twoheadrightarrow B_{dR}/B_{dR}^+$.*
(2) *Ut^{-1} generates $B_{max}^{\phi=1}$ as a sub-$\mathbb{Q}_p$-algebra of B_{dR}.*

Let X be the subalgebra of B_{dR} generated by Ut^{-1} over $\mathbb{Q}_p$. It is enough to proved that

$$X_n = \mathrm{Fil}^{-n} X = X \cap \mathrm{Fil}^{-n} B_{\mathrm{dR}}.$$

Then $\mathrm{gr}X \to \mathrm{gr}B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ is surjective, implying that $X \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ is surjective.

Take $x \in \mathbb{C}_p \cdot t^{-n} = \mathrm{Fil}^{-n} B_{\mathrm{dR}}/\mathrm{Fil}^{-n+1} B_{\mathrm{dR}}$. There exists a $y \in \mathbb{C}_p$ such that $y^n = x \cdot t^n$ and $x = \left(\frac{y}{t}\right)^n$. Then by the proposition, there exists a $v = \log[s]$ for some $s \in U^+$ such that $\theta(v) = \log s^{(0)} = y$.

We conclude that the maps is surjective, since $v^n/t^n \mapsto x$. □

Suppose V is a de Rham representation. Then there is an exact sequence,

$$0 \to V \to B_{\max}^{\phi=1} \otimes_{\mathbb{Q}_p} V \to (B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes_{\mathbb{Q}_p} V \to 0.$$

Take G_K-cohomology. Then in the long exact sequence associated the above short exact sequence, the first nontrivial differential map is

$$\underline{\exp} : H^0(G_K, B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \otimes V) \to H^1(G_K, V).$$

Note that $H^0(G_K, B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \otimes V) = D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V)$.

For example, let F be a commutative formal group of finite height over $\mathcal{O}_K$, where $K/\mathbb{Q}_p$ is a finite extension. Let T be the Tate module of F and let A be the integral closure of U_K in K^{alg}. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T/p^nT & \longrightarrow & F(A) & \xrightarrow{p^n} & F(A) & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow p & & \downarrow id & & \\ 0 & \longrightarrow & T/p^{n+1}T \cdot t & \longrightarrow & U & \longrightarrow & F(A) & \longrightarrow & 0 \end{array}$$

From which we see the exact sequence,

$$0 \to T \to \varprojlim F(A) \to F(A) \to 0$$

where the limit is taken over $x \mapsto px$.

Consider the Bloch-Kato expansion

$$F_K = F \otimes_{\mathcal{O}_K} K$$

There is an exponential

$$\underline{\exp} : TF_K \to F_K(K)$$

for $V = T \otimes \mathbb{Q}_p$.

Lemma 16.10. *There is an isomorphism*

$$TF_K \cong D_{dR}(V)/FilD_{dR}(V)$$

Proof. The morphism $V \otimes H^1_{\mathrm{dR}}(F) \to B_{\mathrm{dR}}$ is non-degenerate and respects the filtration.

$$\begin{array}{ccc} H^1_{\mathrm{dR}}(F) & \supset & \dfrac{\{\text{invariant 1-forms}\}}{\{\text{exact 1-forms}\}} \\ \mathrm{Fil}^0 & & \mathrm{Fil}^1 \end{array}$$

Aside 16.11. A discussion ensued which was hard to follow. The justification relies on the pairing Brent constructed,

$$B_{\mathrm{dR}} \otimes V = \hom_K(H^1_{\mathrm{dR}}, B_{\mathrm{dR}})$$

Take Galois invariants to see we have a filtration

$$\begin{array}{ccc} D_{\mathrm{dR}}(V) & \supset U & \supset 0 \\ \mathrm{Fil}^{-1} & \supset \mathrm{Fil}^0 & \supset \mathrm{Fil}^1 = 0 \end{array}$$

Thus

$$D_{\mathrm{dR}}(V)/U = \left(\mathrm{Fil}^1 H^1_{\mathrm{dR}}(F)\right)^\wedge = (\{\text{invariant 1-forms}\})^\wedge = (T^*F)^\wedge = TF,$$

where the superscript wedge denotes the dual. □

Proposition 16.12. *We have a commutative diagram*

$$\begin{array}{ccc} TF_K & \xrightarrow{\underline{\exp}} & F(\mathcal{O}_K) \otimes \mathbb{Q} \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \partial\otimes\mathbb{Q}} \\ D_{dR}(V)/Fil^0 D_{dR}(V) & \xrightarrow{\exp} & H^1(G_K, V) \end{array}$$

where the partial derivative is the morphism

$$\partial : F(\mathcal{O}_K) = H^0(G_K, F(A)) \to H^1(K, T).$$

Aside 16.13. In the closing discussion, some comments were made. Note the image of the lower Bloch-Kato exponential is the image of the Kummer map. Using the Bloch-Kato exponential, we can construct a nice ??? for an representation. It allows us to compute the dimension of the subspace of all exponentials which are crystalline???

Proof. Let $\chi \in \hom(T, \mathbb{Z}_p(1)) \xrightarrow{\sim} \hom_{f,g}(F, \mathbb{G}_m)$. Then we have a commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & \lim_{\leftarrow} F(A) & \longrightarrow & F(A) & \longrightarrow & 0\ . \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \chi} & & {\scriptstyle \chi}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & U^+ \subset \lim_{\leftarrow} A^\wedge & \longrightarrow & A^\wedge & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \log[]} & & \downarrow{\scriptstyle \log} & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & U = B_{\max}^{\phi=p} \cap B_{\mathrm{dR}}^+ & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mathrm{inj}} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}_p(1) & \longrightarrow & B_{\max}^{\phi=1}(1) & \longrightarrow & (B_{\mathrm{dR}}/B_{\mathrm{dR}}^+)(1) & \longrightarrow & 0
\end{array}$$

We tensor with $T(-1)$ and then with $\mathbb{Q}_p$ to get an exact sequence for T. Let $V = T \otimes \mathbb{Q}_p$. So the bottom line becomes,

$$0 \to V \to B_{\max}^{\phi=1} \otimes \to (B_{\mathrm{dR}}/B_{\mathrm{dR}}^+) \otimes V \to 0.$$

We arrive at a commutative diagram by taking the top and bottom lines of the above large diagram:

$$\begin{array}{ccc}
\otimes F(\mathcal{O}_K) \otimes \mathbb{Q}_p & \xrightarrow{\partial \otimes \mathbb{Q}_p} & H^1(G_K, T) \otimes \mathbb{Q}_p\ , \\
\downarrow{\scriptstyle \log} & & \downarrow \\
D_{\mathrm{dR}}(V)/\mathrm{Fil}^0 D_{\mathrm{dR}}(V) & \xrightarrow{\underline{\exp}} & H^1(G_K, V)
\end{array}$$

From which we conclude the result. □

References

[1] Berger, Laurent. *Galois representations and* (ϕ, Γ)*-modules*, Online notes. Accessed on 26-07-2010 at `http://www.umpa.ens-lyon.fr/~lberger/ihp2010.html`.

[2] Colmez, Pierre. *Espaces de Banach de dimension finie*, Journal of the Inst. of Math. Jussieu (2002) 1(3), 331439.