

PERFECT POWERS IN LINEAR RECURRING SEQUENCES*

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ABSTRACT. P. Corvaja and U. Zannier [4] showed for linear recurring sequences defined by $G_n = a_1\alpha_1^n + \dots + a_t\alpha_t^n$ ($t \geq 2$) with non-zero rational numbers a_i and integral characteristic roots $\alpha_1 > \dots > \alpha_t > 0$ (α_1, α_2 coprime) that the equation $G_n = x^q$ (for $q \geq 2$) has only finitely many solutions $(n, x) \in \mathbb{N}^2$. In this paper we want to use a quantitative version of W. M. Schmidt's Subspace Theorem (due to J.-H. Evertse [5]) to calculate an upper bound for the number of solutions (n, x) . Combining this with an earlier result of I. Nemes and A. Pethő [11] we establish also an upper bound for the number of solutions (n, x, q) .

1. INTRODUCTION

Let A_1, A_2, \dots, A_k and G_0, G_1, \dots, G_{k-1} be algebraic numbers over the rationals and let (G_n) be a k -th order linear recurring sequence given by

$$(1) \quad G_n = A_1 G_{n-1} + \dots + A_k G_{n-k} \quad \text{for } n = k, k+1, \dots$$

Let $\alpha_1, \alpha_2, \dots, \alpha_t$ be the distinct roots of the corresponding characteristic polynomial

$$(2) \quad X^k - A_1 X^{k-1} - \dots - A_k.$$

Then for $n \geq 0$

$$G_n = P_1(n)\alpha_1^n + P_2(n)\alpha_2^n + \dots + P_t(n)\alpha_t^n,$$

where $P_i(n)$ is a polynomial with degree less than the multiplicity of α_i ; the coefficients of $P_i(n)$ are elements of the field: $\mathbb{Q}(G_0, \dots, G_{k-1}, A_1, \dots, A_k, \alpha_1, \dots, \alpha_t)$.

The recurring sequence is called simple, if all characteristic roots are simple. (G_n) is called nondegenerate, if no quotient α_i/α_j for all $1 \leq i < j \leq t$ is equal to a root of unity and degenerate otherwise. Observe that, even if (G_n) is degenerate, there exists a positive integer d such that, (G_{r+md})

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is nondegenerate on each of the d arithmetic progressions with $0 \leq r < d$. Therefore, restricting to nondegenerate recurring sequences causes no substantial loss of generality.

In the present paper we deal with the Diophantine equation

$$(3) \quad G_n = Ex^q, \quad E \in \mathbb{Z} \setminus \{0\}$$

which was earlier investigated by several authors (e.g. cf. [21]).

For the Fibonacci sequence (F_n) , Cohn [2] and Wyler [28], independently, proved that F_n is a square only if $n = 0, 1, 2$ and 13 . Cohn [3] and Steiner [24] solved the equations $F_n = 2x^2$ and $F_n = 3x^2$. They also proved the corresponding results for the Lucas sequence (L_n) . London and Finkelstein [10] determined all cubes in the Fibonacci sequence; Lagarias and Weisser [9] gave another proof. Steiner [23] derived some partial results for higher powers. The proofs of these results do not depend on estimates for linear forms in logarithms. Pethő [14], [15] used the theory of linear forms in logarithms and computer calculations to determine all the cubes and fifth powers in the Fibonacci sequence.

For a nondegenerate recurring sequence (G_n) of order 2 induced by a (rational) integral recurrence, it has been proved, independently, by Pethő [13] and Shorey and Stewart [19] that for the solutions $x \in \mathbb{Z}$, $|x| > 1$ and $q \geq 2$ of (3) $\max(|x|, q, n)$ is bounded by an effectively computable constant depending only on E and the sequence (G_n) . In fact, Pethő proved that $\max(|x|, q, n)$ is bounded by an effectively computable number depending only on the greatest prime divisor of E and on the coefficients and initial values of (G_n) (provided that the coefficients are coprime integers). Pethő extended this result to the equation $G_n = bx^q$ with $b \in S$, where S is a set of integers composed solely of a finite number of primes.

Shorey and Stewart [19] proved the above finiteness result for certain recurring sequences of order > 2 . Let (G_n) be a nondegenerate linear recurring sequence given by

$$(4) \quad G_n = \lambda_1 \alpha_1^n + P_2(n) \alpha_2^n + \cdots + P_t(n) \alpha_t^n,$$

where λ_1 is a non-zero constant, $|\alpha_1| > |\alpha_j|$ for $j = 2, \dots, t$, and $G_n - \lambda_1 \alpha_1^n \neq 0$. Then assuming $x, q > 1$ the solutions q of (3) can be bounded by an effectively computable constant which depends on the coefficients and initial values of the recurrence. Kiss [8] proved that, in fact, q is less than a number which is effectively computable in terms of the greatest prime divisor of E and the coefficients and the initial values of the sequence (G_n) .

Shorey and Stewart [20] considered recurring sequences (G_n) satisfying (4). Assuming that $P_2(n)$ is a non-zero constant, $t = 3$ and $|\alpha_1| = |\alpha_2| = |\alpha_3|$ they showed, that (3) has only finitely many integral solutions E, x, q and n with the greatest prime divisor of E bounded by a given positive integer P , and $|x| > 1, q > 2$ and $n \geq 0$.

Finkelstein [6], Williams [27] and Steiner [24] proved that 1, 2 and 5 are the only Fibonacci numbers of the form $x^2 + 1$. Finkelstein [7] established a similar result for Lucas numbers. Stewart [25] and Shorey and Stewart [20] investigated the equation

$$(5) \quad G_n = x^q + c,$$

where $c \in \mathbb{Z}$ and (G_n) is a simple, nondegenerate second order recurring sequence of rational integers. Assuming $|A_2| = 1$, they showed for the integral solutions of (5) that the maximum of $n \geq 0, |x| > 1$ and $q \geq 3$ is less than an effectively computable constant depending on c , the coefficients and the initial values of the recurrence. In the case $q = 2$ they obtained a similar result under additional technical conditions. Furthermore, Shorey and Stewart [20] proved, that if α and β are multiplicatively independent with one root inside the unit circle, then (5) has only finitely many solutions in integers n, x and q with $n \geq 0, |x| > 1$ and $q > 2$.

Nemes and Pethő [11], [12] studied the more general equation

$$(6) \quad G_n = Ex^q + T(x),$$

where $T(x)$ is a polynomial of degree r and of height H with integral coefficients. For fixed $E \in \mathbb{Z}$ and T they established bounds for the integral solutions n, q, x with $|x|, q > 1$. Let (G_n) be defined as in (4) and assume

$$(7) \quad |\alpha_1| > |\alpha_2| > |\alpha_j|, \quad \text{for } j = 3, \dots, t,$$

with $\alpha_2 \neq \pm 1$. Nemes and Pethő showed that $q < C_1$ provided that $n > C_2$ and $r < C_3q$, where C_1, C_2 and C_3 are suitable positive numbers which are effectively computable in terms of E, H and the coefficients and initial values of the recurrence. Nemes and Pethő were also able to show that if q is a fixed integer larger than one and (6) has infinitely many integral solutions n and x , then $T(x)$ can be characterized in terms of the Chebyshev polynomials. Kiss [8] and Shorey and Stewart [20] dealt with equation (6) for nondegenerate linear recurring sequences (G_n) of arbitrary order, under condition (7) and the additional assumptions that $E = 1$ and d is the degree of α_1 over \mathbb{Q} , α_1 and α_2 are multiplicatively independent and $\alpha_2 \neq \pm 1$. Then they showed

that there are only finitely many integers n, x and q with $n \geq 0, |x| > 1$ and

$$q > \max \left(\frac{d \log |\alpha_1|}{\log(|\alpha_1| / \max(1, |\alpha_2|))}, d + r \right)$$

for which (6) holds.

Recently Corvaja and Zannier [4] considered linear recurrences defined by

$$G_n = a_1 \alpha_1^n + a_2 \alpha_2^n + \cdots + a_t \alpha_t^n,$$

where $t \geq 2, a_1, a_2, \dots, a_t$ are non-zero rational numbers, $\alpha_1 > \alpha_2 > \cdots > \alpha_t > 0$ are integers. They used Schmidt's Subspace Theorem [16], [17] to show that for every integer $q \geq 2$ the equation

$$(8) \quad G_n = x^q$$

has only finitely many solutions $(n, x) \in \mathbb{N}^2$ assuming that G_n is not identically a perfect q th power for any n in a suitable arithmetic progression.

2. RESULTS

Our first main result gives a quantitative version of the above result of Corvaja and Zannier [4].

Theorem 1. *Let (G_n) be a linear recurring sequence defined by*

$$(9) \quad G_n = a_1 \alpha_1^n + a_2 \alpha_2^n + \cdots + a_t \alpha_t^n,$$

where $t \geq 2, a_1, a_2, \dots, a_t$ are non-zero rational numbers, $\alpha_1 > \alpha_2 > \cdots > \alpha_t > 0$ are integers and such that for given $q \geq 2$ there is no $r \in \{0, \dots, q-1\}$ with G_{mq+r} a perfect q th power for all $m \in \mathbb{N}$. Then the number of solutions $(n, x) \in \mathbb{N}^2$ of the equation

$$G_n = x^q$$

is finite and can be bounded above by an explicitly computable number depending on $q, a_1, a_2, \dots, a_t, \alpha_1, \dots, \alpha_t$.

Remark 1. Corvaja and Zannier [4] showed that (9) is the q th power of an integer for infinitely many $n \in \mathbb{N}$, if and only if there exist integers $r \in \{0, \dots, q-1\}, b \geq 1$ and

$$H_n = c_1 \beta_1^n + \cdots + c_s \beta_s^n,$$

where c_1, \dots, c_s are non-zero rational numbers and $\beta_1 > \dots > \beta_s > 0$ are integers as above, such that

$$G_n = b^{n-r} H_n^q.$$

In particular, at least one of the functions $m \mapsto G_{mq+r}$, ($r = 0, \dots, q-1$) is a q th power in the ring of complex functions (with pointwise multiplication) of the form (9), or G_n is a perfect q th power for any n in a suitable

arithmetic progression.

Remark 2. Observe that one can effectively determine whether G_{mq+r} is a perfect q th power or not (see again [4]). A sufficient condition is that α_1, α_2 are coprime.

Remark 3. The example

$$G_n = 18^n + 2 \cdot 6^n + 2^n$$

shows that the condition in Theorem 1 that G_{mq+r} not be a perfect q th power for every m can not be removed. Indeed, in this example the coefficients and roots of G_n satisfy the conditions of Theorem 1, but

$$G_{2m} = (18^m + 2^m)^2,$$

so G_{2m} is a perfect square for all $m \in \mathbb{N}$.

Remark 4. The assumption $\alpha_1 > \alpha_2 > \dots > \alpha_t > 0$ guarantees that the recurring sequence (G_n) is nondegenerate.

Remark 5. We want to mention that the proof of Theorem 1 should also work in the case when (G_n) is a linear recurring sequence with algebraic characteristic roots $\alpha_1, \dots, \alpha_t$, which are multiplicatively independent and satisfy

$$|\alpha_1| > |\alpha_i|, \quad \forall i = 2, \dots, t,$$

and with $a_i \in \mathbb{Q}(\alpha_1, \dots, \alpha_t)$ for all $i = 1, \dots, t$.

Our second main result extends Theorem 1 to the situation when also q is considered to be variable.

Theorem 2. *Let (G_n) be a linear recurring sequence defined by*

$$G_n = a_1 \alpha_1^n + a_2 \alpha_2^n + \dots + a_t \alpha_t^n,$$

where a_1, \dots, a_t ($t \geq 3$) are non-zero rational numbers, $\alpha_1 > \dots > \alpha_t > 0$ are integers and such that (for fixed $q \geq 2$) there is no $r \in \{0, \dots, q-1\}$ with G_{mq+r} a perfect q th power for all $m \in \mathbb{N}$. Then the equation

$$G_n = x^q$$

has only finitely many integral solutions $n, x > 1, q$. The number of solutions can be bounded by an explicitly computable constant C depending only on the recurrence.

Remark 6. The assumption $t \geq 3$ means no loss of generality, because for $t = 2$ this theorem is already well known (see [13], [19]).

Remark 7. Our proof of Theorem 2 depends on an application of the result of Nemes and Pethő [11] which was mentioned in the introduction in detail.

Remark 8. By Remark 5 and the theorem in [11] it should also be possible to obtain the following finiteness result: Let G_n be the n th term of a linear recurrence sequence defined by (9), where $t \geq 3$, $\alpha_1, \dots, \alpha_t$ are multiplicatively independent algebraic numbers with

$$|\alpha_1| > |\alpha_2| > |\alpha_i|, \quad \forall i = 3, \dots, t,$$

and $a_i \in \mathbb{Q}(\alpha_1, \dots, \alpha_t)$ for all $i = 1, \dots, t$. Assuming that for fixed $q \geq 2$ there is no $r \in \{0, \dots, q-1\}$ with G_{mq+r} a perfect q th power for all $m \in \mathbb{N}$, the equation

$$G_n = x^q$$

has only finitely many integral solutions $n, x > 1, q$.

3. AUXILIARY RESULTS

Our proof of Theorem 1 depends on a quantitative version of the Subspace Theorem due to J.-H. Evertse [5].

Let K be an algebraic number field. Denote its ring of integers by O_K and its collection of places by M_K . For $v \in M_K$, $x \in K$, we define the absolute value $|x|_v$ by

- (i) $|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}$ if v corresponds to the embedding $\sigma : K \hookrightarrow \mathbb{R}$;
- (ii) $|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]} = |\bar{\sigma}(x)|^{2/[K:\mathbb{Q}]}$ if v corresponds to the pair of conjugate complex embedding $\sigma, \bar{\sigma} : K \hookrightarrow \mathbb{C}$;

(iii) $|x|_v = (N_{\wp})^{-\text{ord}_{\wp}(x)/[K:\mathbb{Q}]}$ if v corresponds to the prime ideal \wp of O_K . Here $N_{\wp} = \#(O_K/\wp)$ is the norm of \wp and $\text{ord}_{\wp}(x)$ the exponent of \wp in the prime ideal composition of (x) , with $\text{ord}_{\wp}(0) := \infty$. In case (i) or (ii) we call v real infinite or complex infinite, respectively; in case (iii) we call v finite. These absolute values satisfy the *Product formula*

$$(10) \quad \prod_{v \in M_K} |x|_v = 1 \quad \text{for } x \in K^*.$$

The *height* of $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ with $\mathbf{x} \neq \mathbf{0}$ is defined as follows: for $v \in M_K$ put

$$\begin{aligned} |\mathbf{x}|_v &= \left(\sum_{i=1}^n |x_i|_v^{2[K:\mathbb{Q}]} \right)^{1/(2[K:\mathbb{Q}])} && \text{if } v \text{ is real infinite,} \\ |\mathbf{x}|_v &= \left(\sum_{i=1}^n |x_i|_v^{[K:\mathbb{Q}]} \right)^{1/[K:\mathbb{Q}]} && \text{if } v \text{ is complex infinite,} \\ |\mathbf{x}|_v &= \max(|x_1|_v, \dots, |x_n|_v) && \text{if } v \text{ is finite} \end{aligned}$$

(note that for infinite places v , $|\cdot|_v$ is a power of the Euclidean norm). Now define

$$\mathcal{H}(\mathbf{x}) = \mathcal{H}(x_1, \dots, x_n) = \prod_v |\mathbf{x}|_v.$$

For a linear form $l(\mathbf{X}) = a_1 X_1 + \dots + a_n X_n$ with algebraic coefficients we define $\mathcal{H}(l) := \mathcal{H}(\mathbf{a})$, where $\mathbf{a} = (a_1, \dots, a_n)$ and if $\mathbf{a} \in K^n$ then we put $|l|_v = |\mathbf{a}|_v$ for $v \in M_K$. Further we define the number field $K(l) := K(a_1/a_j, \dots, a_n/a_j)$ for any j with $a_j \neq 0$; this is independent of the choice of j .

We are now ready to state Evertse's result [5]. The following notation is used:

- S is a finite set of places on K of cardinality s containing all infinite places;
- $\{l_{1v}, \dots, l_{nv}\}$, $v \in S$ are linearly independent sets of linear forms in n variables with algebraic coefficients such that

$$\mathcal{H}(l_{iv}) \leq H, \quad [K(l_{iv}) : K] \leq D \quad \text{for } v \in S, i = 1, \dots, n.$$

We choose for every place $v \in M_K$ a continuation of $|\cdot|_v$ to the algebraic closure of K and denote this also by $|\cdot|_v$.

Theorem 3. (Quantitative Subspace Theorem, Evertse) *Let $0 < \delta < 1$ and consider the inequality for $\mathbf{x} \in K^n$*

$$(11) \quad \prod_{v \in S} \prod_{i=1}^n \frac{|l_{iv}(\mathbf{x})|_v}{|\mathbf{x}|_v} < \left(\prod_{v \in S} |\det(l_{1v}, \dots, l_{nv})|_v \right) \cdot \mathcal{H}(\mathbf{x})^{-n-\delta}.$$

Then the following assertions hold:

- (i) *There are proper linear subspaces T_1, \dots, T_{t_1} of K^n , with*

$$t_1 \leq (2^{60n^2} \cdot \delta^{-7n})^s \log 4D \cdot \log \log 4D$$

such that every solution $\mathbf{x} \in K^n$ of (11) satisfying $\mathcal{H}(\mathbf{x}) \geq H$ belongs to $T_1 \cup \dots \cup T_{t_1}$.

- (ii) *There are proper linear subspaces S_1, \dots, S_{t_2} of K^n , with*

$$t_2 \leq (150n^4 \cdot \delta^{-1})^{ns+1} (2 + \log \log 2H)$$

such that every solution $\mathbf{x} \in K^n$ of (11) satisfying $\mathcal{H}(\mathbf{x}) < H$ belongs to $S_1 \cup \dots \cup S_{t_2}$.

We also need the following theorem of W. M. Schmidt [18] concerning the zero multiplicity of a nondegenerate recurring sequence.

Theorem 4. (W. M. Schmidt) *Suppose that $(G_n)_{n \in \mathbb{Z}}$ is a nondegenerate linear recurring sequence of complex numbers, whose characteristic polynomial has k distinct roots of multiplicity $\leq a$. Then the number of solutions $n \in \mathbb{Z}$ of the equation*

$$G_n = 0,$$

can be bounded above by

$$c(k, a) = e^{(7k^a)^{8k^a}}.$$

(This number of solutions is called the zero multiplicity of the recurrence.)

In the proof of Theorem 2 we also apply the following result due to A. Baker [1]. Let us first recall the definition of the absolute logarithmic Weil height: For an algebraic number β let $P_\beta(x) = x^k + a_{k-1}x^{k-1} + \dots + a_0 \in \mathbb{Z}[x]$ denote the minimal polynomial of β . Further let $\beta_1 = \beta, \beta_2, \dots, \beta_k$ denote the conjugates of β . Then we call

$$h(\beta) = \frac{1}{k} \log \left(\prod_{i=1}^k \max\{1, |\beta_i|\} \right)$$

the absolute logarithmic Weil height of β .

Theorem 5. (A. Baker) *Let $\alpha_1, \dots, \alpha_k$ be algebraic numbers, different from 0 or 1, $K = \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ and d let be the degree $[K : \mathbb{Q}]$. For $i = 1, \dots, k$ set*

$$h_i = \max \left\{ h(\alpha_i), \frac{e |\log \alpha_i|}{d}, \frac{1}{d} \right\}.$$

Let $b_1, \dots, b_k \in \mathbb{Z}$, $b_k > 0$, $\Lambda = b_1 \log \alpha_1 + \dots + b_k \log \alpha_k \neq 0$ and $B = \max\{2, |b_1|, \dots, |b_{k-1}|\}$. Then we have

$$(12) \quad \log |\Lambda| > -C(k)d^{k+2}h_1 \dots h_k \log (C(k)d^{k+2}h_1 \dots h_{k-1}) \log b_k - \frac{B}{b_k},$$

where

$$C(k) = 2^{26k} k^{3k}.$$

A proof of this theorem with the given explicit constants can be found in the monograph of Waldschmidt [26], page 309, Corollary 9.24.

Below we have collected some simple lemmas which are needed in our proofs.

Lemma 1. *Let $N_{j,k}$ denote the number of formal summands of $(a_1 + \cdots + a_k)^j$, where a_1, \dots, a_k denote formal commuting variables. Then*

$$N_{j,k} = \binom{k+j-1}{j}.$$

This is well known from combinatorics.

Lemma 2. *Let d be a positive integer. Then for the complex function $f(z) = (1+z)^{1/d}$ we have*

$$\left| f(z) - \sum_{k=0}^n \binom{1/d}{k} z^k \right| \leq \frac{1}{d(n+1)(1-|z|)} \cdot |z|^{n+1}$$

for $z \in \mathbb{C}$, $|z| < 1$, where we have chosen the branch of $(1+z)^{1/d}$ which is holomorphic on $\mathbb{C} \setminus (-\infty, -1]$ and which is equal to the positive d -th root of $(1+z)$ for $z \in \mathbb{R}$, $z > -1$.

Proof. It is well-known that for $z \in \mathbb{C}$, $|z| < 1$ we have

$$f(z) = \sum_{k=0}^{\infty} \binom{1/d}{k} z^k.$$

Because of

$$(n+1) \left| \binom{1/d}{n+1} \right| = \frac{\frac{1}{d} \cdot (1 - \frac{1}{d}) \cdot \dots \cdot (n - \frac{1}{d})}{1 \cdot \dots \cdot n} < \frac{1}{d} < 1$$

we obtain

$$\begin{aligned} \left| f(z) - \sum_{k=0}^n \binom{1/d}{k} z^k \right| &\leq \sum_{k=n+1}^{\infty} \left| \binom{1/d}{k} \right| \cdot |z|^k \leq \\ &\leq \frac{1}{d(n+1)} \sum_{k=n+1}^{\infty} |z|^k = \\ &= \frac{1}{d(n+1)(1-|z|)} \cdot |z|^{n+1}, \end{aligned}$$

and therefore the proof is complete. \square

Lemma 3. *Let $a, b \geq 0$ and let $x \in \mathbb{R}$ be the largest solution of $x = a + b \log x$. If $b > e^2$ then*

$$x < 2(a + b \log b).$$

This lemma is due to A. Pethő and B.M.M. de Weger [22].

4. PROOF OF THEOREM 1

According to Theorem 4 the number of solutions of (8) of the form $(n, 0)$, $n \in \mathbb{N}$ can be estimated by

$$c_1(t) \leq e^{(7t)8t}.$$

Therefore we can restrict ourselves to solutions of the form $(n, x) \in \mathbb{N}^2$ with $x \neq 0$. These solutions are denoted by $(n, x_n) \in \mathbb{N}^2$ with $n \in \Sigma$, where Σ is a set of positive integers.

Let us now consider the expansion of the function $f(z) = (1+z)^{1/q}$ around the origin

$$(1+z)^{1/q} = \sum_{j=0}^{\infty} \binom{1/q}{j} z^j, \quad \text{with } |z| \leq 1, z \neq -1.$$

We approximate $G_n^{1/q}$ by defining

$$H_m := (a_1 \alpha_1^r)^{1/q} \cdot \alpha_1^m \left[1 + \sum_{j=1}^R \binom{1/q}{j} \cdot \left(\sum_{i=2}^t \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}} \right)^j \right],$$

where $R \geq 1$ is an integer to be chosen later and where we have set $n = mq + r$ with $n \in \mathbb{N}$, $r \in \{0, \dots, q-1\}$. We write

$$H_m = \sum_{i=1}^h d_i \left(\frac{e_i}{b} \right)^m,$$

where $d_i \in \mathbb{Q} \left((a_1 \alpha_1^r)^{1/q} \right)^*$, e_i, b are integers, $b > 0$, and the e_i/b are non-zero distinct rational numbers. Clearly, H_m is nondegenerate (the roots are all positive) and we have

$$[\mathbb{Q} \left((a_1 \alpha_1^r)^{1/q} \right) : \mathbb{Q}] \leq q.$$

By Lemma 1 we obtain

$$h \leq \binom{R+t-1}{R}.$$

On the other hand, we have

$$\left| \sum_{i=2}^t \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}} \right| \leq (t-1) c \left(\frac{\alpha_2}{\alpha_1} \right)^{mq+r} \leq \frac{1}{2} < 1$$

where

$$c := \max \left\{ \left| \frac{a_i}{a_1} \right| \mid i = 2, \dots, t \right\},$$

if

$$(13) \quad m \geq \frac{\log 2(t-1)c}{q \log \frac{\alpha_1}{\alpha_2}}.$$

Therefore, by Lemma 2 for m we get for m large enough

$$\begin{aligned} |H_m - x_{mq+r}| &= |G_{mq+r}^{1/q} - H_m| \leq \\ &\leq \left| a_1^{1/q} \right| \cdot \alpha_1^{r/q} \cdot \alpha_1^m \cdot \frac{1}{q(R+1) \left(1 - \left| \sum_{i=2}^t \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}} \right| \right)} \left| \sum_{i=2}^t \frac{a_i \alpha_i^{mq+r}}{a_1 \alpha_1^{mq+r}} \right|^{R+1} \leq \\ &\leq \left| a_1^{1/q} \right| \cdot \alpha_1 \cdot \alpha_1^m \cdot \frac{2}{q(R+1)} \left[(t-1)c \left(\frac{\alpha_2}{\alpha_1} \right)^r \left(\frac{\alpha_2}{\alpha_1} \right)^{mq} \right]^{R+1} \leq \\ &\leq \left| a_1^{1/q} \right| \cdot \alpha_1^{r/q} \cdot [(t-1)c]^{R+1} \cdot \alpha_1^m \cdot \left(\frac{\alpha_2}{\alpha_1} \right)^{mq(R+1)}. \end{aligned}$$

Thus we derive

$$(14) \quad |H_m - x_{mq+r}| \leq c_2(R) \cdot l_1^m$$

where we have set

$$l_1 := \alpha_1 \cdot \left(\frac{\alpha_2}{\alpha_1} \right)^{q(R+1)}$$

and

$$c_2(R) := \left| a_1^{1/q} \right| \cdot \alpha_1 \cdot [(t-1)c]^{R+1}.$$

Now choose

$$(15) \quad R > \max \left\{ 1, \frac{1}{q} \frac{\log \alpha_1}{\log \frac{\alpha_1}{\alpha_2}} - 1 \right\}.$$

Then $0 < l_1 < 1$. Put

$$l := \frac{l_1 + 1}{2}.$$

Then for m large enough, to be more precise, for

$$(16) \quad m > \frac{\log c_2(R)}{\log \frac{l}{l_1}}$$

we have

$$c_2(R) \cdot l_1^m = c_2(R) \left(\frac{l_1}{l} \right)^m \cdot l^m < l^m.$$

Consequently we obtain

$$(17) \quad |H_m - x_{mq+r}| < l^m \quad \text{with } 0 < l < 1,$$

provided that R satisfies (15) and m satisfies (13) and (16).

Now let S be the set of places of \mathbb{Q} consisting of ∞ and all primes dividing some of the e_i or b . Extend each place in S to $K := \mathbb{Q}((a_1\alpha_1^r)^{1/q})$ in some way, the infinite place being extended such that it coincides with the complex absolute value in the given embedding of K in \mathbb{C} . Define the linear forms $L_{i,v}$ for $v \in S$ and $i = 1, \dots, h$ as follows: $L_{0,\infty} := L := X_0 - \sum_{i=1}^h d_i X_i$, $L_{i,\infty} := X_i$ for $i = 1, \dots, h$, while for $v \in S$, $v \neq \infty$, put $L_{i,v} := X_i$ for $i = 1, \dots, h$. Then $\{L_{0,v}, \dots, L_{h,v}\}$, $v \in S$ are linearly independent sets of linear forms in $h+1$ variables with coefficients in K . Furthermore we have

$$\mathcal{H}(L_{i,v}) = \begin{cases} \mathcal{H}(1, -d_1, \dots, -d_h) =: \tilde{H} & \text{for } i = 0, v = \infty \\ \mathcal{H}(0, \dots, 0, 1, 0, \dots, 0) = 1 & \text{else.} \end{cases}$$

We set $H := \max\{1, \tilde{H}\}$ then it follows $\mathcal{H}(L_{i,v}) \leq H$, for $v \in S$, $i = 0, \dots, h$. Then $\mathbb{Q}(L_{i,v}) = \mathbb{Q}$, for $v \neq \infty$, $[\mathbb{Q}(L_{0,\infty}) : \mathbb{Q}] \leq q$ and therefore

$$[\mathbb{Q}(L_{i,v}) : \mathbb{Q}] \leq q \quad \forall v \in S, i = 0, \dots, h.$$

For $n \in \Sigma$ define the vector $\mathbf{x}_m = (b^m x_{mq+r}, e_1^m, \dots, e_h^m) \in \mathbb{Z}^{h+1}$.

From (17) we obtain $|L_{0,\infty}(\mathbf{x}_m)| \leq (bl)^m$. Recall that S includes all primes dividing b and that the x_{mq+r} are integers. Thus by the product formula (10),

$$\prod_{v \in S \setminus \{\infty\}} |L_{0,v}(\mathbf{x}_m)|_v = \prod_{v \in S \setminus \{\infty\}} |b^m x_{mq+r}|_v \leq \prod_{v \in S \setminus \{\infty\}} |b^m|_v = b^{-m}.$$

Moreover, since S includes also the primes dividing the numbers e_i the product formula (10) gives

$$\prod_{v \in S} \prod_{i=1}^h |L_{i,v}(\mathbf{x}_m)|_v = \prod_{v \in S} \prod_{i=1}^h |e_i^m|_v = 1.$$

Thus we obtain

$$\prod_{v \in S} \prod_{i=0}^h \frac{|L_{i,v}(\mathbf{x}_m)|_v}{|\mathbf{x}_m|_v} \leq \left(\prod_{v \in S} |\mathbf{x}_m|_v \right)^{-h-1} \cdot l^m.$$

Since the coordinates of the vectors \mathbf{x}_m are integers we have $|\mathbf{x}_m|_v \leq 1$ for $v \in M_{\mathbb{Q}} \setminus \{\infty\}$. Further, we have

$$|\mathbf{x}_m|_{\infty} \leq A^m$$

for some real A independent of m . Indeed, we have

$$|x_{mq+r}| \leq |x_{mq+r} - H_m| + |H_m| \leq l^m + h \cdot \tilde{c} \cdot a^m \leq 1 + h \cdot \tilde{c} \cdot a^m \leq \tilde{a}^m,$$

with

$$\begin{aligned} \tilde{c} &:= \max\{|d_i| \mid i = 1, \dots, h\}, \\ a &:= \max\left\{\left|\frac{e_i}{b}\right| \mid i = 1, \dots, h\right\}, \end{aligned}$$

and $\tilde{a} := (1 + h \cdot \tilde{c})(1 + a)$. Hence

$$\begin{aligned} |\mathbf{x}_m|_\infty &= \left(|b^m x_{mq+r}|^2 + \sum_{i=1}^h |e_i|^2 \right)^{1/2} \leq \\ &\leq ((b\tilde{a})^{2m} + h(ba)^{2m})^{1/2} \leq A^m \end{aligned}$$

with $A := (h + 1)\tilde{a}b$. It follows that

$$\mathcal{H}(\mathbf{x}_m) = \prod_{v \in M_{\mathbb{Q}}} |\mathbf{x}_m|_v \leq \prod_{v \in S} |\mathbf{x}_m|_v \leq |\mathbf{x}_m|_\infty \leq A^m.$$

Lastly we have

$$\det(L_{0,v}, \dots, L_{h,v}) = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & 1 & 0 & \cdots & 0 \\ * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & 0 & 0 & \cdots & 1 \end{vmatrix} = 1,$$

which yields

$$|\det(L_{0,v}, \dots, L_{h,v})|_v = 1 \quad \forall v \in S.$$

Combining our estimates we get

$$\prod_{v \in S} \prod_{i=0}^h \frac{|L_{i,v}(\mathbf{x}_m)|_v}{|\mathbf{x}_m|_v} < \left(\prod_{v \in S} |\det(L_{0,v}, \dots, L_{h,v})|_v \right) \cdot \mathcal{H}(\mathbf{x}_m)^{-h-1-\delta}$$

for all m with (16), provided that $\delta < \log(1/l)/\log A$. By Theorem 3 there exist finitely many non-zero rational linear forms $\Lambda_1(X_0, \dots, X_h), \dots, \Lambda_g(X_0, \dots, X_h)$ with

$$\begin{aligned} g &\leq (2^{60(h+1)^2} \cdot \delta^{-7(h+1)})^s \log 4q \cdot \log \log 4q + \\ &\quad + (150(h+1)^4 \cdot \delta^{-1})^{(h+1)s+1} (2 + \log \log 2H), \end{aligned}$$

such that each vector \mathbf{x}_m is a zero of some Λ_j .

Suppose first Λ_j does not depend on X_0 . Then, if $\Lambda_j(\mathbf{x}_m) = 0$, we have a nontrivial relation

$$\sum_{i=1}^h u_i \left(\frac{e_i}{b}\right)^m = 0, \quad u_i \in \mathbb{Q}, \quad i = 1, \dots, h.$$

By Theorem 4 this can hold for at most a finite number of m . More precisely, the number of solutions m can be estimated by

$$c_1(h) = e^{(7h)^{8h}},$$

since (H_m) is nondegenerate.

Suppose that Λ_j depends on X_0 and that $\Lambda_j(\mathbf{x}_m) = 0$. Then we have

$$(18) \quad \sum_{i=1}^h v_i \left(\frac{e_i}{b}\right)^m = x_{mq+r}, \quad v_i \in \mathbb{Q}, \quad i = 1, \dots, h.$$

Set

$$U_m := \sum_{i=1}^h v_i \left(\frac{e_i}{b}\right)^m,$$

then U_m is a nondegenerate, simple recurring sequence and we obtain

$$U_m^q = x_{mq+r}^q = G_{mq+r}.$$

Hence

$$V_m := \left(\sum_{i=1}^h v_i \left(\frac{e_i}{b}\right)^m \right)^q - \sum_{i=1}^t a_i \alpha_i^r (\alpha_i^q)^m,$$

has the form

$$V_m = \sum_{i=1}^p b_i \beta_i^m$$

with $b_i \in \mathbb{Q}$, $\beta_i \in \mathbb{Q}^+$, $i = 1, \dots, p$. Therefore V_m is a nondegenerate, simple recurring sequence, and we conclude by Lemma 1

$$p \leq t + \binom{h+q-1}{q} \leq t + \binom{\binom{R+t-1}{t} + q - 1}{q}.$$

Observe that by our assumptions $V_m = 0$ does not hold for every m hence an i with $b_i \neq 0$ exists. Again by Theorem 4 we can bound the number of solutions of (18) by

$$c_1(p) = e^{(7p)^{8p}}.$$

Therefore the number of solutions of (8) can be estimated by

$$\begin{aligned} \tilde{C}(q) &:= \\ &= e^{(7t)^{8t}} + \frac{\log 2(t-1)c}{\log \frac{\alpha_1}{\alpha_2}} + q \cdot \left[\left\{ (2^{60(\tilde{h}+1)^2} \cdot \delta^{-7(\tilde{h}+1)})^s \cdot \right. \right. \\ &\quad \cdot \log 4q \cdot \log \log 4q + (150(\tilde{h}+1)^4 \cdot \delta^{-1})^{(\tilde{h}+1)^{s+1}} \cdot (2 + \log \log 2H) \left. \right\} \cdot \\ &\quad \cdot \left. \left\{ e^{(7\tilde{h})^{8\tilde{h}}} + e^{(7\tilde{p})^{8\tilde{p}}} \right\} + \frac{\log c_2(R)}{\log \frac{l}{l_1}} \right], \end{aligned}$$

where

$$\begin{aligned} \tilde{h} &= \binom{R+t-1}{R}, \quad \tilde{p} = \binom{\tilde{h}+q-1}{q} + t \\ H &= \max\{1, \mathcal{H}(1, -d_1, \dots, -d_h)\} \\ s &= |S|, \\ c_2(R) &= \left| a_1^{1/q} \right| \cdot \alpha_1 \cdot [(t-1)c]^{R+1}, \\ \delta &< \log(1/l) / \log A, \end{aligned}$$

and l is as in (17). □

5. PROOF OF THEOREM 2

Assume that $n, x > 1, q \geq 2$ is a solution of (8) and write

$$x^q = G_n = a_1 \alpha_1^n + B(n).$$

We distinguish two cases:

Case 1: $B(n) = 0$.

Here we get

$$|B(n)| = \left| a_2 \alpha_2^n \left(1 + \sum_{i=3}^t \frac{a_i}{a_2} \left(\frac{\alpha_i}{\alpha_2} \right)^n \right) \right| \geq |a_2| \alpha_2^n \left| 1 - \sum_{i=3}^t \frac{a_i}{a_2} \left(\frac{\alpha_i}{\alpha_2} \right)^n \right| > 0,$$

since

$$\begin{aligned} &\left| \sum_{i=3}^t \frac{a_i}{a_2} \left(\frac{\alpha_i}{\alpha_2} \right)^n \right| \leq \\ &\leq \max \left\{ \left| \frac{a_i}{a_2} \right| \mid i = 3, \dots, t \right\} \cdot (t-2) \cdot \left(\frac{\alpha_3}{\alpha_2} \right)^n \leq \\ &\leq tc |a_2|^{-1} \left(\frac{\alpha_3}{\alpha_2} \right)^n < 1, \end{aligned}$$

where $c = \max\{|a_i| \mid i = 1, \dots, t\}$, whenever

$$n > \frac{\log(tc|a_2|^{-1})}{\log \frac{\alpha_2}{\alpha_3}} =: n_1.$$

Therefore $n \leq n_1$ must hold and we obtain from $a_1\alpha_1^n = x^q$ and $x \geq 2$

$$q = \frac{\log(|a_1|\alpha_1^n)}{\log x} \leq \frac{\log(c\alpha_1^{n_1})}{\log 2}.$$

Case 2: $B(n) \neq 0$.

In this case we first set

$$\delta := \frac{1}{2} \left(1 - \frac{\log \alpha_2}{\log \alpha_1} \right).$$

Then we get

$$|B(n)| \leq tc\alpha_2^n < \frac{1}{2}\alpha_1^{n(1-\delta)},$$

if

$$n > \frac{2 \log(2ct)}{\log \frac{\alpha_1}{\alpha_2}} =: n_2.$$

Further

$$(19) \quad \frac{x^q}{a_1\alpha_1^n} = 1 + \frac{B(n)}{a_1\alpha_1^n},$$

so

$$(20) \quad 1 - (|a_1|\alpha_1^{\delta n})^{-1} \leq |a_1|^{-1}\alpha_1^{-n}x^q \leq 1 + (|a_1|\alpha_1^{\delta n})^{-1},$$

where we have used that $(|a_1|\alpha_1^{\delta n})^{-1} < \frac{1}{2}$, if

$$n > \frac{\log(2|a_1|^{-1})}{\delta \log \alpha_1} =: n_3.$$

Taking logarithms and using the inequalities $|\log(1+x)| \leq x$ and $|\log(1-x)| \leq 2x$ for $0 \leq x < \frac{1}{2}$, we derive by (20)

$$-2|a_1|^{-1}\alpha_1^{-\delta n} \leq -\log |a_1| - n \log \alpha_1 + q \log x \leq 2|a_1|^{-1}\alpha_1^{-\delta n}.$$

Thus

$$(21) \quad |-\log |a_1| - n \log \alpha_1 + q \log x| \leq 2|a_1|^{-1}\alpha_1^{-\delta n}.$$

Put $\Lambda = -\log |a_1| - n \log \alpha_1 + q \log x$. From (19) and the fact that $B(n) \neq 0$, we get $\Lambda \neq 0$. Thus we can employ Theorem 5 and obtain for $n \geq 2$

$$(22) \quad |\Lambda| > \exp \left\{ -C(3)h_1h_2 \log(C(3)h_1h_2) e \log x \log q - \frac{n}{q} \right\},$$

where $C(3) = 2^{78}3^9$ and

$$\begin{aligned} h_1 &= \max\{h(|a_1|^{-1}), e|\log|a_1||, 1\}, \\ h_2 &= \max\{h(\alpha_1), e\log\alpha_1, 1\} = e\log\alpha_1. \end{aligned}$$

Set

$$c_3 := C(3)h_1h_2\log(C(3)h_1h_2)e.$$

A comparison of (21) and (22) reveals that

$$(23) \quad -c_3\log q\log x - \frac{n}{q} < \log(2|a_1|^{-1}) - n\delta\log\alpha_1.$$

However, for $n > \max\{n_1, n_2\}$,

$$\frac{1}{2}|a_1|\alpha_1^n \leq |a_1|\alpha_1^n - |B(n)| \leq x^q \leq |a_1|\alpha_1^n + |B(n)| \leq ct\alpha_1^n.$$

Thus, for

$$n > \max\left\{\frac{\log ct}{\log\alpha_1}, \frac{2\log(2|a_1|^{-1})}{\log\alpha_1}\right\} =: n_4$$

we obtain

$$\frac{\log\alpha_1}{2}n < q\log x < 2\log\alpha_1 n.$$

Writing this as

$$\frac{n}{q} < \frac{2\log x}{\log\alpha_1}, \quad \frac{\log x}{2\log\alpha_1}q < n,$$

(23) can be reformulated as

$$q < \frac{2\log(\frac{1}{2}|a_1|^{-1})}{\delta\log 2} + \frac{4}{\delta\log\alpha_1} + \frac{2c_3}{\delta}\log q.$$

Thus by Lemma 3

$$q < 2\left(\frac{2\log(\frac{1}{2}|a_1|^{-1})}{\delta\log 2} + \frac{4}{\delta\log\alpha_1} + \frac{2c_3}{\delta}\log\left(\frac{2c_3}{\delta}\right)\right) =: C_1,$$

if $n > \max\{2, n_2, n_3, n_4\} =: n_5$. Otherwise, we have

$$q \leq \frac{\log(ct\alpha_1^{n_5})}{\log 2}.$$

Altogether we derive

$$q \leq \max\left\{\frac{\log(ct\alpha_1^c)}{\log 2}, C_1(q)\right\} =: C,$$

where $\bar{c} := \max\{2, n_1, n_2, n_3, n_4\}$. For the number of solutions $n, x > 1, q \geq 2$ of (8) we finally obtain the upper bound

$$\sum_{q=2}^{\bar{c}} \tilde{C}(q),$$

and therefore the proof is finished. \square

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