DIVISIBILITY PROPERTIES OF HYPERGEOMETRIC POLYNOMIALS

CLEMENS FUCHS[†] AND T.N. SHOREY

Abstract. In this paper we give effective upper bounds for the degree k of divisors (over $\mathbb{Q}$) of hypergeometric polynomials defined by

$$\sum_{j=0}^{n} a_j \frac{(a)_j}{(b)_j (c)_j} x^j,$$

where $(m)_j = m(m+1)\cdots(m+j-1)$ denotes the Pochhammer symbol and $a_0, \ldots, a_n$ are integers with $|a_0| = |a_n| = 1, a = -n-r, b = \alpha+1, c \geq 1$ and $\alpha = -tn-s-1, tn+s$ for integers $r \geq 0, t \geq 1, s, c$ bounded in terms of k. These results generalize on earlier results of the authors and others on generalized Laguerre polynomials.

2000 *Mathematics Subject Classification:* 11C08, 33C45, 11N05.
Keywords: Hypergeometric polynomials, irreducibility, p-adic Newton polygons, gaps between primes.

1. Introduction and results

For rational numbers a, b, c the hypergeometric polynomials are defined by

$$g_{a,b,c}(x) = \sum_{j=0}^{n} \frac{(a)_j}{(b)_j (c)_j} x^j,$$

where $(m)_j = m(m+1)\cdots(m+j-1)$ denotes the Pochhammer symbol. We mention that such polynomials appear by truncating the infinite series given by generalized hypergeometric functions of type ${}_2F_2(a,1;b,c;x)$ (with the usual notation for such functions). For $a = -n, b = \alpha+1, c = 1$ one gets

$$\begin{aligned} g_{-n,\alpha+1,1}(x) &= \frac{n!}{(\alpha+1)\cdots(\alpha+n)} \sum_{j=0}^{n} \frac{(\alpha+n)\cdots(\alpha+j+1)}{(n-j)!j!}(-x)^j \\ &= \frac{n!}{(\alpha+1)\cdots(\alpha+n)} L_n^{(\alpha)}(x) \end{aligned}$$

the generalized Laguerre polynomials (up to a constant).

[†]Corresponding author.

Let n, s and t be integers with $n \geq 2$ and $|s| \leq n$ and

$$\alpha = -tn - s - 1 \text{ with } t \geq 2 \tag{1}$$

or

$$\alpha = tn + s \text{ with } t \geq 1. \tag{2}$$

Such polynomials have been studied extensively, especially the case $L_n^{(\alpha)}(x)$, starting with work by Schur [10, 11], Coleman [1] and Filaseta and others (see e.g. [3]). We also mention the papers [2] and [5] since we shall be using their arguments. Now we are additionally assuming that

$$a = -n - r, \quad b = \alpha + 1, \quad c \geq 1 \tag{3}$$

for integers r, c with $r \geq 0$. Let α satisfy (1). Then

$$g_{a,b,c}(x) = \frac{(n+r)!(c-1)!}{((t-1)n+s+1)\cdots(tn+s)(n+r+c-1)!} \sum_{j=0}^{n} c_j x^{n-j}$$

with

$$c_j = \binom{n+r+c-1}{r+j}((t-1)n+s+1)\cdots((t-1)n+s+j) \tag{4}$$

and therefore we have for $m \in \{0, \ldots, n\}$ that

$$\frac{c_n}{c_{n-m}} = \frac{(tn+s)!}{(tn+s-m)!} \frac{(n+r-m)!}{(n+r)!} \frac{(m+c-1)!}{(c-1)!}. \tag{5}$$

Now let α satisfy (2). Then we have

$$g_{a,b,c}(x) = \frac{(-1)^n(c-1)!(n+r)!}{(tn+s+1)\cdots((t+1)n+s)(n+c+r-1)!} \sum_{j=0}^{n} c'_j x^{n-j}$$

with

$$c'_j = (-1)^j \binom{n+r+c-1}{r+j}((t+1)n+s-j+1)\cdots((t+1)n+s) \tag{6}$$

and therefore we have

$$\frac{c'_n}{c'_{n-m}} = (-1)^m \frac{(tn+s+m)!}{(tn+s)!} \frac{(n+r-m)!}{(n+r)!} \frac{(m+c-1)!}{(c-1)!} \tag{7}$$

for $m \in \{0, \ldots, n\}$. For $0 \leq j \leq n$, we write $d_j = c_j$ or c'_j according as α satisfies (1) or (2). Moreover, we set

$$f(x) = \sum_{j=0}^{n} d_j x^{n-j}$$

and

$$F(x) = \sum_{j=0}^{n} a_j d_j x^{n-j}$$

for integers $a_0, \ldots, a_n$. Here we notice that $F(x)$ is the polynomial stated in the abstract.

Our intention is to generalize the results of [6] to this extended setting. It was proved there that for integers $a_0, \ldots, a_n$ with $|a_0| = |a_n| = 1$ there exist effectively computable absolute constants η_0 and ε such that for all $\eta_0 < k \le \frac{n}{2}$ and for all α with $t < \varepsilon \log k, 0 \le s < \varepsilon k \log k$ the polynomial $F(x)$ does not have a factor of degree k. We also mention that for $2 \le k \le \frac{n}{2}$, it was proved in [9, Theorem 1.3] that if for given $\varepsilon > 0$ the hypergeometric polynomial $g_{-n-r,\alpha+1,c}$ with $0 \le \alpha \le k$ and $r+c < (1/3-\varepsilon)k$ has a divisor of degree k, then k is bounded by an effectively computable constant depending only on ε, and in [9, Theorem 1.4] that $g_{-n,\alpha+1,1}$ with $\alpha = -n-s-1$ and $0 \le s \le 0.95k$ has no factor of degree k at all.

In the sequel we will denote by $\eta_1, \eta_2, \ldots$ effectively computable absolute positive real constants.

Theorem 1. *Let $a_0, \ldots, a_n$ be any integers with $|a_0| = |a_n| = 1$. Then there exist constants $\varepsilon > 0$ and η_1 such that for all $\eta_1 < k \le \frac{n}{2}$ and for all α satisfying* (1) *with $t \ge 4$ or* (2) *with $t \ge 3$ and for*

$$t < \varepsilon \log k, \quad \max\{r, c\} < k, \quad |s| < \varepsilon k \vartheta,$$

where $\vartheta = \log k$, the polynomial $F(x)$ does not have factor of degree k.

Moreover, under the abc-conjecture, the statement holds true with $\vartheta = \log n$.

For small values of t in both the cases for α we also get results, but under slightly stronger restrictions. We state them separately in the following theorem.

Theorem 2. *The statement of Theorem 1 holds true for*

$$\max\{r, c, |s|\} < k,\ r + c + |s| < n^{6/11+\varepsilon} \quad \text{if } \alpha \text{ satisfies (1) with } t = 2,$$
$$\max\{r, c, |s|\} < k,\ r + c < n^{6/11+\varepsilon} \quad \text{if } \alpha \text{ satisfies (2) with } t = 1,$$

and for

$$\max\{r, c\} < k, \quad |s| < \varepsilon k \vartheta, \quad r + c < n^{6/11+\varepsilon}$$

if α satisfies (1) *with $t = 3$ or* (2) *with $t = 2$.*

The two theorems imply [6, Theorem 1,2] apart from the values of ε. Further we observe that we cover the negative values of s in contrast to the situation in [6].

In the proof we will again use the p-adic Newton polygon, where the prime p satisfies certain properties. Let us write v_p for the p-adic valuation and $v_p(0) = \infty$. Then we use the following lemma, which we take from [2]:

Lemma 1. *Let k and l be integers with $k > l \geq 0$ and $k \leq \frac{n}{2}$. Suppose that*

$$g(x) = \sum_{j=0}^{n} b_j x^{n-j} \in \mathbb{Z}[x]$$

and p is a prime such that $p \nmid b_0, p|b_j$ for all $j \in \{l+1, \ldots, n\}$ and the slope of the right-most edge of the Newton polygon for $f(x)$

$$\max_{1 \leq m \leq n} \left\{ \frac{v(b_n) - v(b_{n-m})}{m} \right\}$$

is $< 1/k$. Then for any integers $a_0, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial

$$G(x) = \sum_{j=0}^{n} a_j b_j x^{n-j}$$

cannot have a factor with degree in the interval $[l+1, k]$.

The existence of such primes is the main challenge (and also the most significant difference to our results in [6]) and this will be guaranteed by tools from analytic number theory. The result on primes that we are needing is the following lemma:

Lemma 2. *There exists a constant η_2 such that for all $x > \eta_2$ and for all $\frac{6}{11} < \theta \leq 1$ we have*

$$0.969 \frac{y}{\log x} \leq \pi(x) - \pi(x-y)$$

for $y = x^\theta$, where $\pi(x)$ is the prime counting function.

This result is taken from [7]. Moreover, for the conditional result in Theorem 1 we recall the abc-conjecture that we will use.

Lemma 3 (abc-Conjecture). *For every $\epsilon > 0$ there exists a constant $\gamma = \gamma(\epsilon)$ depending only on ϵ such that for all coprime nonzero integers a, b, c with $a + b = c$ the inequality*

$$\max\{|a|, |b|, |c|\} < \gamma N(abc)^{1+\epsilon}$$

holds, where $N(m)$ denotes the product over all different prime divisors of m.

Now we have everything ready to give the proof of Theorem 1 and Theorem 2 that will be done simultaneously in the next section.

2. Proof of Theorem 1 and 2

For the proof we assume that $F(x)$ has a factor of degree k such that $k \leq \frac{n}{2}$ and k exceeds a sufficiently large constant η_1. Let $\vartheta = \log n$ if the abc-conjecture holds and $\vartheta = \log k$ otherwise. Moreover, we put $\delta = 1/4$.

By Lemma 2 there exists ℓ with

$$n^{13/22} \leq \ell < ((t+1)n+s)^{13/22}$$

such that $(t-1)n+s+\ell$ or $(t+1)n+s-\ell+1$ is a prime p according as (1) or (2) holds, respectively. Then it follows from (4) and (6), respectively, that $p \| d_j$ for $j \in \{\ell, \ldots, n\}$ (here we use, as usual, $d \| d_j$ for $d | d_j$ and $d^2 \nmid d_j$). Next we show that $p > n+c+r$. For this we have to take special care of the small values of t. We have

$$\begin{aligned}
(t-1)n+s+\ell &\geq n-|s|+n^{6/11+\varepsilon} > n+c+r && \text{if (1) with } t=2,\\
(t-1)n+s+\ell &\geq n+n^{6/11+\varepsilon} > n+c+r && \text{if (1) with } t=3,\\
(t+1)n+s-\ell+1 &> n+(n/2-n^{7/11})+1 &&\\
&> n+n^{6/11+\varepsilon} > n+c+r && \text{if (2) with } t=1,\\
(t+1)n+s-\ell+1 &> n+(n-n^{7/11}) &&\\
&> n+n^{6/11+\varepsilon} > n+c+r && \text{if (2) with } t=2
\end{aligned}$$

and finally $p > 2n \geq n+c+r$ in all other cases. This implies $p \nmid d_0$. Therefore, the right-most edge of the p-adic Newton polygon for $f(x)$ has slope $< 1/k$. By Lemma 1 we conclude that $k \leq \ell \leq (3\varepsilon n \log n)^{13/22} \leq n^{7/11}$.

Now we will first consider the case (1), i.e. that $\alpha = -tn-s-1$. We write $z = 6\varepsilon k\vartheta$. Observe that every prime $p > z \geq k$ that divides $((t-1)n+s+1)\cdots((t-1)n+s+k)$ divides exactly one of the factors, so $p|(t-1)n+s+1+\ell$ for some $0 \leq \ell \leq k-1$. We shall show that a prime with this property exists. For this purpose we use the following lemma (cf. [4, Lemma 6] and [6, Lemma 5]).

Lemma 4. *Let z be a positive real number. For each prime $p \leq z$, let $d_p \in \{n, n-1, \ldots, n-k+1\}$ with $v_p(d_p)$ maximal. Define*

$$Q_z = Q_z(n,k) = \prod_{p>z} p^{v_p(A)}$$

with $A = n(n-1)\cdots(n-k+1)$. Then

$$Q_z \geq \frac{n(n-1)\cdots(n-k+1)}{(k-1)!\prod_{p\leq z} p^{v_p(d_p)}} \geq \frac{(n-k+1)^{k-\pi(z)}}{(k-1)!},$$

where $\pi(z)$ denotes the number of primes $\leq z$.

By the above lemma we get for $\vartheta = \log k$ that

$$\begin{aligned} Q_z((t-1)n+s+k,k) &\geq \frac{((t-1)n+s+1)^k}{(k-1)!((t-1)n+s)^{\pi(z)}} \geq n^{k-2\pi(z)-7k/11} \\ &\geq n^{(4/11-12\varepsilon(1+\delta)^2)k} > 1, \end{aligned}$$

where we have used the inequality $(k-1)! \leq k^k \leq n^{7k/11}$ and the estimate

$$\pi(z) \leq \frac{(1+\delta)6\varepsilon k\vartheta}{\log(6\varepsilon k\vartheta)} \leq 6\varepsilon(1+\delta)^2 k,$$

that follows at once from the prime number theorem. It remains to show that we also have a prime $p > \eta_3 k \log n > z$ for some η_3 and for ε small enough, dividing $((t-1)n+s+1)\cdots((t-1)n+s+k)$, if we assume the abc-conjecture to be true. For this we just have to follow the arguments of [8, Theorem 1]. We give the proof for the readers convenience (and since the statement that is proved there, at first sight, does not seem to be connected to what we need). For a prime p dividing two different factors of this product of k consecutive terms we have $p \leq k$. Thus

$$\prod_{i=1}^{k} N((t-1)n+s+i) \leq \left(\prod_{p\leq P} p\right) \prod_{p\leq k} p^{\lfloor k/p \rfloor} \leq \eta_4 \exp\left(\eta_5(P + k\log k)\right),$$

where P denotes the largest prime divisor of $((t-1)n+s+1)\cdots((t-1)n+s+k)$ and $N(m)$ the product over all primes dividing m. Now let j_1, j_2 with $N((t-1)n+s+j_1) \leq N((t-1)n+s+j_2)$ be the smallest two values in the set $\{N((t-1)n+s+j); 1 \leq j \leq k\}$. It follows

$$\begin{aligned} N((t-1)n+s+j_2) &\leq \left(\prod_{i=1}^{k} N((t-1)n+s+i)\right)^{1/(k-1)} \\ &= \exp\left(\eta_6(P/k + \log k)\right). \end{aligned}$$

We apply Lemma 3 with $\epsilon = 1$ to the equation

$$\frac{(t-1)n+s+j_1}{d} - \frac{(t-1)n+s+j_2}{d} = \frac{j_1-j_2}{d}$$

and get

$$\begin{aligned} n &\leq \eta_7 \left(N\left(\frac{(t-1)n+s+j_1}{d}\right) N\left(\frac{(t-1)n+s+j_2}{d}\right) \frac{|j_1-j_2|}{d} \right)^2 \\ &\leq \exp\left(\eta_8 (P/k + \log k)\right), \end{aligned}$$

where d denotes the greatest common divisor of $(t-1)n+s+j_1$ and $(t-1)n+s+j_2$. Finally, this implies $P > \eta_9 k \log n$.

Thus there is a prime $p > z$ dividing $((t-1)n+s+1)\cdots((t-1)n+s+k)$, say p divides $(t-1)n+s+1+\ell$ with $0 \leq \ell \leq k-1$. We may assume that $p \nmid n+c+i$ for every $0 \leq i \leq r-1$, since assuming the contrary we have $p|n+c+i$, which implies that p divides $|(t-1)n+s+1+\ell-(t-1)(n+c+i)| \leq |s|+1+\ell+tc+tr \leq \varepsilon k\vartheta + k + 2\varepsilon k \log k \leq 4\varepsilon k\vartheta \leq z$, a contradiction. It follows that p satisfies $p|c_j$ for $\ell+1 \leq j \leq n$ and $p \nmid c_0$. Define $m = m(p) \in \{1,\ldots,n\}$ such that

$$\frac{v_p(c_n)-v_p(c_{n-m(p)})}{m(p)} = \max_{1\leq m\leq n}\left\{\frac{v_p(c_n)-v_p(c_{n-m})}{m}\right\}$$

is the slope of the right most edge of the p-adic Newton polygon for $f(x)$ with respect to p. Then by Lemma 1 and (5) we conclude

$$\begin{aligned} \frac{1}{k} &\leq \frac{v_p(c_n)-v_p(c_{n-m})}{m} \\ &\leq \frac{1}{m}\left[v_p\left(\frac{(tn+s)!}{(tn+s-m)!}\right) - v_p\left(\binom{n+r}{m}\right) + v_p\left(\binom{m+c-1}{c-1}\right)\right]. \end{aligned}$$

For estimating the third summand we may assume that $m > 5\varepsilon k\vartheta$, since otherwise $m+c-1 \leq 6\varepsilon k\vartheta < p$ and so this summand is zero, and therefore we get

$$\begin{aligned} \frac{1}{m} v_p\left(\binom{m+c-1}{c-1}\right) &\leq \frac{1}{m} v_p((m+c-1)!) \leq \frac{m+c-1}{m(p-1)} \\ &= \frac{1}{p-1} + \frac{c-1}{m(p-1)} < \frac{1}{5\varepsilon k\vartheta} + \frac{k}{5\varepsilon k\vartheta 5\varepsilon k\vartheta} \leq \frac{1}{4k}. \end{aligned}$$

If p does not divide $(tn+s-m+1)\cdots(tn+s)$ then we immediately get a contradiction. Thus we may assume that p divides $tn+s-i$ with $0 \leq i \leq m-1$. But then it also divides $t((t-1)n+s+\ell+1)-(t-1)(tn+s-i) = t(\ell+1)+s+(t-1)i$ and therefore $p \leq 2\varepsilon k\theta + \varepsilon m \log k \leq 2\varepsilon k\vartheta + 2\varepsilon m \log k$.

Since $p > z = 6\varepsilon k\vartheta$, this implies that

$$\frac{2k\vartheta}{\log k} < m. \tag{8}$$

Moreover we get

$$\begin{aligned}\frac{3}{4k} \le \frac{1}{m} v_p\left(\frac{(tn+s)!}{(tn+s-m)!}\right) &\le \frac{1}{m}\sum_{j=1}^{\infty}\left(\left\lfloor\frac{tn+s}{p^j}\right\rfloor - \left\lfloor\frac{tn+s-m}{p^j}\right\rfloor\right)\\ &\le \frac{1}{m}\sum_{j=1}^{J}\left(\frac{m}{p^j}+1\right) \le \frac{1}{p-1}+\frac{J}{m} \le \frac{1}{12k}+\frac{J}{m},\end{aligned}$$

where

$$J := \left\lfloor \frac{\log(tn+s)}{\log p}\right\rfloor.$$

This gives $m \le 3kJ/2$ and thus

$$m \le \frac{3k}{2}\frac{\log(tn+s)}{\log p} < \frac{3(1+\delta)k\log n}{2\log k}. \tag{9}$$

If $\vartheta = \log n$, then it follows from (8) and (9) that $2\log n = 2\vartheta < \frac{3}{2}(1+\delta)\log n$. This contradiction proves the result if we assume that the abc-conjecture is true. Now we give the proof for the case $\vartheta = \log k$. In fact all the above statements are true for every prime $p > z$ dividing $(t-1)n + s + 1 + \ell$ for some $0 \le \ell \le k-1$. Especially this is the case for the inequalities (8) and (9). Next we will prove the existence of such a prime with even stronger assumptions.

Let U be the set of numbers $(t-1)n+s+1+j$ with $0 \le j \le k-1$, where for every prime $q \le z$ we have removed those numbers $d_q \in \{(t-1)n + s+1, \ldots, (t-1)n+s+k-1\}$ with $v_q(d_q)$ maximal. We mention that all elements of U are $\ge n$ if $t > 2$ and $\ge n/2$ if $t = 2$. Now let Ω be the set of all primes $q > z$ with $v_q(u) > 0$ for some $u \in U$ and $q^{v_q(u)} \le (2k+m)\varepsilon\log k$ for all $u \in U$. Observe that all such q divide exactly one $u \in U$, since $q > z \ge k$. Thus we have

$$\begin{aligned}\log\left(\prod_{u\in U}\prod_{q\in\Omega} q^{v_q(u)}\right) &\le \log\left(\prod_{z<q\le(2k+m)\varepsilon\log k}(2k+m)\varepsilon\log k\right)\\ &\le \pi((2k+m)\varepsilon\log k)\log((2k+m)\varepsilon\log k) \le (1+\delta)\varepsilon(2k+m)\log k\\ &\le \varepsilon(1+\delta)(2k\log k + 3k\log n) \le 5\varepsilon(1+\delta)k\log n,\end{aligned}$$

where for the second summand (9) was used. It follows that

$$\log\left(\prod_{u\in U}\prod_{q\le z} q^{v_q(u)}\right)+\log\left(\prod_{u\in U}\prod_{q\in\Omega} q^{v_q(u)}\right)$$
$$\le \frac{(1+\delta)6\varepsilon k\log k}{\log(6\varepsilon k\log k)}\log k+5\varepsilon(1+\delta)k\log n\le\frac{2}{3}k\log n,$$

since $k\le n^{7/11}$. On the other side we have

$$\log\left(\prod_{u\in U} u\right)\ge\log(((t-1)n+s+1)\cdots((t-1)n+s+k-\pi(z)))$$
$$\ge (k-\pi(z))\log\frac{n}{2}>\frac{2}{3}k\log n.$$

By comparing the lower and upper bound just obtained we conclude that there is a prime $q>z$ that divides some element $u\in U$ with the additional property that $q^{v_q(u)}>(2k+m)\varepsilon\log k$. We write $u=(t-1)n+s+\ell+1, 0\le \ell\le k-1$ and define f by $q^{f-1}\le(2k+m)\varepsilon\log k<q^f$ and such that $q^f|u$. Observe that $1\le f\le v_q(u)$.

Now if q^f divides $tn+s-i$ for some $0\le i\le m-1$, then it also divides $|t((t-1)n+s+\ell+1)-(t-1)(tn+s-i)|\le t\ell+t+|s|+(t-1)i<3\varepsilon k\log k$ which contradicts the fact that $q^f>(2k+m)\varepsilon\log k$ by (8). Thus q^f does not divide $tn+s-i$ for any $0\le i\le m-1$ and we conclude

$$\frac{3}{4k}\le\frac{1}{m}v_q\left(\frac{(tn+s)!}{(tn+s-m)!}\right)\le\frac{1}{m}\sum_{j=1}^{f-1}\left(\frac{m}{q^j}+1\right)\le\frac{1}{q-1}+\frac{f-1}{m}.$$

For $f=1$ we immediately get a contradiction. For $f\ge 2$ we get

$$2(2k+m)\varepsilon\log k\ge q^{f-1}+(2k+m)\varepsilon\log k\log k\ge(f-1)6\varepsilon k\log k+4\varepsilon k\log k,$$

where we have used (8), and therefore $3(f-1)k<m$, which gives

$$\frac{3}{4k}\le\frac{1}{q-1}+\frac{f-1}{m}<\frac{5}{12k}+\frac{1}{3k}=\frac{3}{4k},$$

a contradiction again. This completes the proof in this case.

Now we come to the case (2), i.e. that $\alpha=tn+s$. Here we can argue in almost the same way. We have to consider primes $p>z=6\varepsilon k\vartheta$ that divide $((t+1)n+s-k+1)\cdots((t+1)n+s)$ and we show by following the arguments from above that such a prime exists. As before we may assume that a prime dividing $(t+1)n+s-\ell$ for some $0\le\ell\le k-1$ does not divide any of $n+c+i$ for $0 \le i \le r-1$, since otherwise we have that p divides $|(t+1)n+s-\ell-(t+1)(n+c+i)| \le |s|+\ell+(t+1)(c+r) \le \varepsilon k\vartheta + k + 4\varepsilon k\log k \le 6\varepsilon k\vartheta = z$. Therefore it follows that such a p satisfies $p|c_j'$ for $\ell+1 \le j \le n$ and $p \nmid c_0'$. Proceeding as in the previous case we conclude from Lemma 1 and (7) that

$$\begin{aligned}\frac{1}{k} \le & \frac{v_p(c_n') - v_p(c_{n-m}')}{m} \\ & \le \frac{1}{m}\left[v_p\left(\frac{(tn+s+m)!}{(tn+s)!}\right) - v_p\left(\binom{n+r}{m}\right) + v_p\left(\binom{m+c-1}{c-1}\right)\right].\end{aligned}$$

In the same way as before we can estimate the third summand and we may assume that p divides $tn+s+m-i$ with $0 \le i \le m-1$ and therefore $6\varepsilon k\vartheta = z < p \le |t((t+1)n+s-\ell)-(t+1)(tn+s+m-i)| \le t\ell+|s|+(t+1)m \le 2\varepsilon k\vartheta + 2m\varepsilon\log k$, which again implies $2k\vartheta/\log k < m$. On the other hand, one shows $m \le 3kJ/2$ with

$$J := \left\lfloor \frac{\log((t+1)n+s)}{\log p} \right\rfloor,$$

which gives $m < 3(1+\delta)k\log n/(2\log k)$. For $\vartheta = \log n$ we conclude the proof by comparing the lower and upper bound for m. Thus we may assume that $\vartheta = \log k$. By arguing as above we get a prime $q > z$ that divides exactly one element of the form $u = (t+1)n+s-\ell, 0 \le \ell \le k-1$ and with $q^{v_q(u)} > (2k+m)\varepsilon\log k$ (observe that now all such elements u are $\ge n$). By defining f as before we conclude that q^f does not divide $tn+s+1+i$ for any $0 \le i \le m-1$, since otherwise it would divide $t((t+1)n+s-\ell)-(t+1)(tn+s+1+i) = -t\ell-ts-(t+1)(1+i)$ that contradicts $q^f > z$ being large. Similar as in the case (1) the proof can be finished.

3. Acknowledgment

This paper was partly written during mutual visits of the authors at their home institutions, to which they are indepted for the hospitality and for financial support.

References

[1] R.F. Coleman, On the Galois groups of the exponential Taylor polynomials, *L'Enseignement Math.* **33** (1987), 183-189.

[2] M. Filaseta, The irreducibility of all but finitely many Bessel polynomials, *Acta Math.* **174** (1995), 383-397.

[3] M. Filaseta, C. Finch, and J.R. Leidy, T.N. Shorey's influence in the theory of irreducible polynomials, In: *Diophantine Equations* (N. Saradha, ed.), Narosa (2008), 77-102.

[4] M. Filaseta, T. Kidd, and O. Trifonov, Laguerre polynomials with Galois group A_m for each m, Preprint, 2008.

[5] M. Filaseta and R.L. Williams Jr., On the irreducibility of a certain class of Laguerre polynomials, *J. Number Theory* **100** (2003), 229-250.

[6] C. Fuchs and T.N. Shorey, Divisibility properties of generalized Laguerre polynomials, *Indag. Math. (N.S.)* **20(2)** (2009), 217-231.

[7] S.T. Lou and Q. Yao, A Chebychev's type of prime number theorem in a short interval. II, *Hardy-Ramanujan J.* **15** (1992), 1-33.

[8] F. Luca On factorials which are products of factorials, *Math. Proc. Camb. Philos. Soc.* **143** (2007), 533-542.

[9] T.N. Shorey and R. Tijdeman, Generalizations of some irreducibility results by Schur, *Acta Arith.*, to appear.

[10] I. Schur, Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen, I., *Sitzungsber. Preuss. Akad. Wiss. Berlin. Phys.-Math. Kl.* **14** (1929), 125-136.

[11] I. Schur, Affektlose Gleichungen in der Theorie der Lagurreschen und Hermiteschen Polynome, *J. Reine Angew. Math.* **165** (1931), 52-58.

Clemens Fuchs
Department of Mathematics, ETH Zurich
Rämistrasse 101, 8092 Zürich, Switzerland
Email: clemens.fuchs@math.ethz.ch

T.N. Shorey
School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, 400005 Mumbai, India
Email: shorey@math.tifr.res.in