DIVISIBILITY PROPERTIES OF GENERALIZED LAGUERRE POLYNOMIALS

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Abstract. In this paper we give effective upper bounds for the degree k of divisors (over $\mathbb{Q}$) of generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, i.e. of

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!},$$

for $\alpha = -tn - s - 1$ and $\alpha = tn + s$ with $t, s \in \mathbb{N}, t = O(\log k)$, $s = O(k \log k)$ and k sufficiently large.

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1. Introduction and results

The generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ are well-known and extensively studied objects in different areas of mathematics, e.g. in analysis, combinatorics and mathematical physics. We define

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!} = \sum_{j=0}^{n} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!j!}(-x)^j$$

for $\alpha \in \mathbb{R}, n \in \mathbb{N}$. An important instance is given by the case $\alpha = -2n-1$, since there we have the relation

$$(-1)^n n! L_n^{(-2n-1)}(x) = x^n B_n(2/x)$$

and $B_n(x)$ are the Bessel polynomials. The leading coefficient is given by $(-1)^n/n!$.

In this paper we are concerned with questions on the divisors of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$. In 1929 I. Schur proved that $L_n^{(0)}(x)$ and $L_n^{(-n-1)}(x)$ are irreducible (over $\mathbb{Q}$) and in 1931 he proved the same for $L_n^{(1)}(x)$ (cf. [14, 15]). A new proof for the case $\alpha = -n-1$ was given by R.F. Coleman [1] in 1987 by using the Newton polygon. In fact Coleman and M. Filaseta developed a new

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method for attacking these kind of problems. An overview of the results can be found in [4]. Filaseta [3] proved in 1995 that $L_n^{(-2n-1)}(x)$ is irreducible for all but finitely many n, and in turn he immediately obtained the same result for the Bessel polynomials (by the connection given above). Later in 2002 Filaseta and O. Trifonov [6] extended this result to all integers n. In the same year, Filaseta and T.-Y. Lam [5] proved that $L_n^{(\alpha)}(x)$ is irreducible for all but finitely n for a fixed $\alpha \in \mathbb{Q}\backslash\mathbb{Z}^-$. We point out that the exclusion of the negative integers is really necessary, since for $\alpha = -r$ with $r \in \mathbb{N}$ an easy computation shows that $L_n^{(\alpha)}(x)$ is reducible for $n \geq r$. The irreducibility for $\alpha = -n - r$ and $r = 2, \ldots, 9$ was shown by F. Hajir [9, 10] (the case $r = 3$ is by E.A. Sell [16]). Hajir also proved that for a fixed positive integer r the polynomial $L_n^{(-n-r)}(x)$ is irreducible for all but finitely many n. We mention that the statements on all but finitely many n are effective in the sense that an explicit lower bound for the n from which onward the statement is true can be given. Hajir conjectured in [10] that for all non-negative integers n, s the generalized Laguerre polynomial $L_n^{(-n-s-1)}(x)$ is irreducible. Another instance of this problem was recently considered by Filseta, T. Kidd and Trifonov in [7]. They showed that $L_n^{(n)}(x)$ is irreducible for every n with $n \equiv 2 \pmod 4$ with the exception $n = 2$, where this is false, and for all other n the polynomial $L_n^{(n)}(x)$ is either irreducible or it is a linear polynomial times an irreducible polynomial of degree $n - 1$.

Instead of proving that $L_n^{(-n-s-1)}(x)$ is irreducible one can try to exclude divisors of large degree for many values of s. Recently, R. Tijdeman and the second author [13] proved that for every $0 \leq \alpha \leq 30, \alpha \in \mathbb{Z}$ and $4 < k \leq \frac{n}{2}$, the polynomial $L_n^{(\alpha)}(x)$ has no factor of degree k. Moreover, they proved that if $2 \leq k \leq \frac{n}{2}$ and s is an integer with $0 \leq s \leq 0.8k$, then $L_n^{(-n-s-1)}(x)$ does not have a factor of degree k. In their paper they study in fact a more general situation than the case of Laguerre polynomials, where the above results are just some special cases. In this paper we shall consider an analogous problem for $L_n^{(t'n+s')}$ where s' and $|t'| \geq 1$ are integers.

Now we fix some notation which we shall always follow in this paper without reference. Let n, s, t be integers with $n \geq 2, 0 \leq s \leq n$ and let α be given by either

$$\alpha = -tn - s - 1 \text{ with } t \geq 2 \tag{1}$$

or

$$\alpha = tn + s \text{ with } t \geq 1. \tag{2}$$

First let α satisfy (1). We have

$$\begin{aligned} L_n^{(\alpha)}(x) &= \sum_{j=0}^{n} \frac{(n - tn - s - 1) \cdots (j + 1 - tn - s - 1)}{(n-j)!j!} (-x)^j \\ &= \sum_{j=0}^{n} \frac{(-(t-1)n - s - 1) \cdots (j - tn - s)}{(n-j)!j!} (-x)^j \\ &= (-1)^n \sum_{j=0}^{n} \frac{((t-1)n + s + 1) \cdots ((t-1)n + s + j)}{(n-j)!j!} x^{n-j} \end{aligned}$$

and therefore

$$(-1)^n n! L_n^{(\alpha)}(x) =: \sum_{j=0}^{n} c_j x^{n-j}$$

where

$$\begin{aligned} c_j &= \binom{n}{j} ((t-1)n + s + 1) \cdots ((t-1)n + s + j) \\ &= \binom{(t-1)n + s + j}{j} (n - j + 1) \cdots n. \end{aligned} \tag{3}$$

Observe that for every $m \in \{0, \ldots, n\}$ we thus have

$$\frac{c_n}{c_{n-m}} = \frac{(tn+s)!}{(tn+s-m)!} \cdot \frac{(n-m)!}{n!} \cdot m!. \tag{4}$$

Let α satisfy (2). Then

$$(-1)^n n! L_n^{(\alpha)}(x) = (-1)^n n! \sum_{j=0}^{n} \binom{(t+1)n + s}{n-j} \frac{(-x)^j}{j!} =: \sum_{j=0}^{n} c'_j x^{n-j},$$

where

$$c'_j = (-1)^j \binom{n}{j} ((t+1)n + s - j + 1) \cdots ((t+1)n + s). \tag{5}$$

Thus

$$\frac{c'_n}{c'_{n-m}} = (-1)^m \frac{(tn+s+m)!}{(tn+s)!} \cdot \frac{(n-m)!}{n!} \cdot m! \tag{6}$$

for every $m \in \{0, 1, \ldots, n\}$. The relations (4) and (6) will be of importance later on. For $0 \leq j \leq n$, we write $d_j = c_j$ or c'_j according as α satisfies (1) or (2), respectively. Moreover, we set

$$f(x) := \sum_{j=0}^{n} d_j x^{n-j}$$

and

$$F(x) := \sum_{j=0}^{n} a_j d_j x^{n-j}$$

for integers $a_0, a_1, \ldots, a_n$. In the sequel we will denote by $\eta_1, \eta_2, \ldots$ effectively computable absolute positive real constants. We have the following result.

Theorem 1. *Let $\varepsilon = 1/5$ if $\alpha < 0$ and $\varepsilon = 1/112$ if $\alpha > 0$. Let $a_0, a_1, \ldots, a_n$ be any integers with $|a_0| = |a_n| = 1$. Then there exists a constant η_1 such that for all k with*

$$\eta_1 < k \leq \frac{n}{2}$$

and for all α satisfying (1) *or* (2) *with*

$$t < \varepsilon \log k, \quad s < \varepsilon k \log k,$$

the polynomial $F(x)$ does not have a factor of degree k.

Concerning the role of $F(x)$ compared to $f(x)$ we mention that many of the results from the introduction are also of such a general shape. As a special case we immediately get the following result for generalized Laguerre polynomials, which we state separately.

Theorem 2. *Let ε be as in Theorem 1. There exists a constant η_2 such that for all k with*

$$\eta_2 < k \leq \frac{n}{2}$$

and all α satisfying (1) *or* (2) *with*

$$t < \varepsilon \log k, \quad s < \varepsilon k \log k,$$

the polynomial $L_n^{(\alpha)}(x)$ has no factor of degree k.

In the proof we will see that if $n \leq k^{42/23}$, then Theorem 1 is also valid with $t < \varepsilon \log n, s < \varepsilon k \log n$. The assumption $n \leq k^{42/23}$ is relaxed as follows in the case of negative integers, i.e. in case α satisfies (1).

Theorem 3. *Let $\varepsilon = 1/7$ and $a_0, a_1, \ldots, a_n$ be any integers with $|a_0| = |a_n| = 1$. There is a constant η_3 such that for all $n > \eta_3$ and all α satisfying* (1) *with*

$$\exp\left((\log n)^{\frac{2}{3}(1+\varepsilon)}\right) \leq k \leq \frac{n}{2}$$

and

$$t < \varepsilon \log n, \quad s < \varepsilon k \log n,$$

the polynomial $F(x)$ does not have a factor of degree k.

The proofs of Theorems 1 and 3 split in two parts. First by using p-adic arguments, especially the p-adic Newton polygon, we reduce the problem to finding a prime p having certain properties. By considering several cases depending on k the proof will be finished.

In the next section we introduce the Newton polygon with respect to a prime p and give some auxiliary results on this polygon, as well as on primes in certain intervals. Afterwards, we will give the proofs of Theorem 1 and 3, respectively.

2. Newton polygons and preliminaries on primes

For a prime p let v_p be a p-adic valuation, i.e. for a positive integer n we have that $v_p(n)$ is the largest integer such that $p^{v_p(n)}|n$ (we will also use the notation $p^{v_p(n)}\|n$, for short) and $v_p(0) = \infty$. We shall also write v for v_p when it will be clear from the context which p we are taking. Let $g(x) = \sum_{j=0}^{n} b_j x^{n-j} \in \mathbb{Z}[x]$ with $b_0 b_n \neq 0$. The p-adic Newton polygon (or just Newton polygon) for $g(x)$ with respect to the prime p is now defined as the polygonal path formed by the lower edges of the convex hull of the points

$$(0, v(b_0)), (1, v(b_1)), (2, v(b_2)), \ldots, (n-1, v(b_{n-1})), (n, v(b_n)).$$

The left-most endpoint is $(0, v(b_0))$ and the right-most endpoint is $(n, v(b_n))$. Moreover, the endpoints of each edge belong to the above set and the slopes of the edges strictly increase from left to right.

Then Filaseta [3, Lemma 2] proved the following result.

Lemma 1. *Let k and ℓ be integers with $k > \ell \geq 0$ and $k \leq n/2$. Suppose that*

$$g(x) = \sum_{j=0}^{n} b_j x^{n-j} \in \mathbb{Z}[x]$$

and p is a prime such that $p \nmid b_0, p|b_j$ for all $j \in \{\ell+1, \ldots, n\}$ and the right-most edge of the Newton polygon for $g(x)$ with respect to p has slope $< 1/k$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial

$$G(x) = \sum_{j=0}^{n} a_j b_j x^{n-j}$$

cannot have a factor with degree in the interval $[\ell+1, k]$.

We apply this lemma in the case $L_n^{(\alpha)}(x)$, in fact we will use $g(x) = f(x)$ and $G(x) = F(x)$.

The next result is an estimate on the difference between consecutive primes and it will be used in the proof of Theorem 1.

Lemma 2 (S.T. Lou and Q. Yao, [12]). *Denote by p_n the n-th prime number and let $\varepsilon > 0$. There exists a constant η_4 such that*

$$p_{n+1} - p_n \leq \eta_4\, p_n^{6/11+\varepsilon}.$$

We remark that the upper bound is already quite good, since under the Riemann hypothesis the ideal bound would be $\ll p_n^{1/2} \log p_n \ll p_n^{1/2+\varepsilon}$ (with the usual meaning of $\ll$). Furthermore, observe that the lemma implies that there is η_5 such that for any $x > \eta_5$ there exists a prime in the interval $[x, x+x^{1/2+1/21}]$ as well as in the interval $[x-x^{1/2+1/21.5}, x] \supseteq [x-x^{47/86}, x]$.

For Theorem 3 we need a result on the largest prime factor in a product of consecutive integers.

Lemma 3 (M. Jutila, [11]). *Let u and k be positive integers and let $P(u,k)$ be the largest prime factor of $(u+1)\cdots(u+k)$. Then there are constants η_6, η_7 and η_8 such that for*

$$k^{3/2} \leq u \leq k^{\eta_6 (\log k)^{1/2}/\log\log k}$$

we have

$$P(u,k) \geq \eta_8 k^{1+\eta_7 \Lambda(k,u)},$$

where $\Lambda(k,u) = (\log k/\log u)^2$.

Now we are ready to prove our assertions. This will be done in the next two sections.

3. Proof of Theorem 1

Let $\varepsilon = 1/5$ if $\alpha < 0$ and $\varepsilon = 1/112$ if $\alpha > 0$ and set $\delta = 1/16$. Let η_1 be sufficiently large. Assume that $F(x)$ has a factor of degree k such that $\eta_1 < k \leq \frac{n}{2}$ and $\varepsilon \log k > t, \varepsilon k \log k > s$. Observe that $n \geq 2k$ and therefore n exceeds a sufficiently large effectively computable absolute constant. We divide the proof of Theorem 1 in two parts according to (1) or (2).

3.1. **The case of negative indices, i.e. the case** (1)**.** First by Lemma 2 (see also the remark made afterwards) it follows that there is a prime p of the form $p = (t-1)n + s + \ell$ with

$$0 < \ell \leq ((t-1)n + s)^{1/2+1/21} \leq (tn)^{23/42} < (n \log n)^{23/42}. \tag{7}$$

From the definition of the c_j (cf. (3)) it follows at once that $p \| c_j$ for $j \in \{\ell, \ldots, n\}$ and $p \nmid c_0 = 1$. Thus the right-most edge of the Newton polygon for $f(x)$ with respect to p has endpoints $(\ell - 1, 0), (n, 1)$ and therefore, by (7), its slope is $1/(n - \ell + 1) < 2/n \leq 1/k$. By Lemma 1 we get a contradiction unless $k \leq \ell$. Hence we can assume $k \leq \ell \leq (tn)^{23/42}$.

Now we consider two different cases depending on the size of k, namely $n^{11/21} \leq k$ and $k < n^{11/21}$:

Assume that $n^{11/21} \leq k$. Then since $k \leq (tn)^{23/42}$ and $1/2 < 11/21$, we have $n^{1/2} < n^{11/21} \leq k \leq (tn)^{23/42}$. We start by proving the following lemma which extends Filaseta [3, Lemma 4] with $t = 2, s = 0$.

Lemma 4. *Let n, r and $k \leq n/2$ be positive integers, $\ell \in \{0, 1, \ldots, k-1\}$ and let $p \geq tk + s + 1$ be a prime number satisfying $p^r \| n - \ell$ and*

$$\frac{\log(tn + s)}{p^r \log p} + \frac{1}{p-1} \leq \frac{1}{k}. \tag{8}$$

Then $F(x)$ does not have a factor with degree $\in [\ell + 1, k]$.

Proof. We start by introducing the function

$$a(n, j) := \left\lfloor \frac{n}{p^j} \right\rfloor - \left\lfloor \frac{n-m}{p^j} \right\rfloor$$

(obviously $a(n, j)$ also depends on m and p), which is equal to the number of multiples of p^j in $(n - m, n]$ and where $\lfloor x \rfloor$ is defined to be the largest integer $\leq x$. Since

$$v(m!) = \sum_{j=1}^{\infty} \left\lfloor \frac{m}{p^j} \right\rfloor < \sum_{j=1}^{\infty} \frac{m}{p^j} = \frac{m}{p-1},$$

we show that it suffices to prove that

$$a(tn + s, j) - a(n, j) \leq \frac{m}{p^r} \tag{9}$$

for all

$$j \leq J := \left\lfloor \frac{\log(tn + s)}{\log p} \right\rfloor.$$

Because of (4) we now have

$$\begin{aligned}\frac{v(c_n)-v(c_{n-m})}{m} &= \frac{v(m!)}{m}+\frac{1}{m}\left[v\left(\frac{(tn+s)!}{(tn+s-m)!}\right)-v\left(\frac{n!}{(n-m)!}\right)\right]\\ &= \frac{v(m!)}{m}+\sum_{j=1}^{\infty}\frac{a(tn+s,j)-a(n,j)}{m},\end{aligned}$$

where the sum in fact just runs over $j \leq J$, since for $j > J$ we have $p^j > tn+s$, which implies $a(tn+s,j) = a(n,j) = 0$. Thus, by using the inequalities obtained so far together with the hypothesis in (8), the slope of the right-most edge of the Newton polygon for $f(x)$ is bounded by

$$\max_{1\leq m\leq n}\left\{\frac{v(c_n)-v(c_{n-m})}{m}\right\} < \frac{1}{p-1}+\frac{\log(tn+s)}{p^r\log p} \leq \frac{1}{k}.$$

Since $p|n(n-1)\cdots(n-\ell)$ and $p > k \geq j$ it follows that $p|c_j$ for $j \in \{\ell+1,\ldots,n\}$ and $p \nmid c_0 = 1$ (cf. equation (3)). Therefore, we get by Lemma 1 that $F(x)$ does not have a factor with degree $\in [\ell+1,k]$. This proves the assertion.

We have to prove (9) and we do it by considering three cases depending on the size of j and m.

Case 1. Assume that $1 \leq j \leq r$: We will use that $p^r \| n-\ell$, which implies that there is an integer u with $n = p^r u + \ell$. It follows that

$$\begin{aligned}a(tn+s,j) &= \left\lfloor\frac{tn+s}{p^j}\right\rfloor-\left\lfloor\frac{tn+s-m}{p^j}\right\rfloor\\ &= \left\lfloor tp^{r-j}u+\frac{t\ell+s}{p^j}\right\rfloor-\left\lfloor tp^{r-j}u+\frac{t\ell+s-m}{p^j}\right\rfloor\\ &= \left\lfloor\frac{t\ell+s}{p^j}\right\rfloor-\left\lfloor\frac{t\ell+s-m}{p^j}\right\rfloor = -\left\lfloor\frac{t\ell+s-m}{p^j}\right\rfloor\\ a(n,j) &= \left\lfloor\frac{n}{p^j}\right\rfloor-\left\lfloor\frac{n-m}{p^j}\right\rfloor = \left\lfloor p^{r-j}u+\frac{\ell}{p^j}\right\rfloor-\left\lfloor p^{r-j}u+\frac{\ell-m}{p^j}\right\rfloor\\ &= \left\lfloor\frac{\ell}{p^j}\right\rfloor-\left\lfloor\frac{\ell-m}{p^j}\right\rfloor = -\left\lfloor\frac{\ell-m}{p^j}\right\rfloor,\end{aligned}$$

where we have used that our prime p satisfies $p^j \geq p > t\ell+s$. Therefore we get

$$a(tn+s,j)-a(n,j) = -\left\lfloor\frac{t\ell+s-m}{p^j}\right\rfloor+\left\lfloor\frac{\ell-m}{p^j}\right\rfloor \leq 0.$$

It follows that (9) is trivially true.

Case 2. Assume now $m \le t\ell + s$: It first follows that $(tn + s - m, tn + s] \subseteq (tn + s - t\ell - s, tn + s] = (tn - t\ell, tn + s]$. Since $p|n - \ell$ we get $p|tn-t\ell$. Together with $p > t\ell+s$ this implies that there is no multiple of p^j in $(tn+s-m, tn+s]$ at all, which gives $a(tn+s, j) = 0$. Thus (9) again holds.

Case 3. Finally assume $j > r, m > t\ell + s$: We observe that the number of multiples of p^r in $(tn + s - m, tn + s] \supset [tn - t\ell, tn + s]$ is bounded by $\lfloor m/p^r \rfloor + 1$. Since $p \ge tk+s+1 > t$, the multiple $tn-t\ell$ of p^r is not divisible by p^j. Therefore

$$a(tn + s, j) \le a(tn + s, r) - 1 \le \left\lfloor \frac{m}{p^r} \right\rfloor .$$

So in this case the inequality (9) holds too.

Altogether, we have covered all cases and this completes the proof of Lemma 4. □

Thus we just have to prove that there is a prime $p > 2\varepsilon k \log k$, which implies $p > (t + 1)k > (t + 1)n^{1/2}$ and $p \ge tk + s + 1$, that divides $n(n - 1) \cdots (n - k + 1)$. Then

$$\frac{\log(tn + s)}{p^r \log p} + \frac{1}{p-1} \le \frac{\log((t + 1)n)}{(t + 1)k \log((t + 1)n^{1/2})} + \frac{1}{(t + 1)k} \le \frac{1}{k},$$

since $\log((t + 1)n)/\log((t + 1)n^{1/2}) \le 2 \le t$ and then the contradiction follows by Lemma 4. To show that such a prime exists we use the following lemma, which is based on an argument first given by Erdős in [2]. We take the following version that can be found in [7, Lemma 6].

Lemma 5. *Let z be a positive real number. For each prime $p \le z$, let $d_p \in \{n, n - 1, \ldots, n - k + 1\}$ with $v_p(d_p)$ maximal. Define*

$$Q_z = Q_z(n, k) = \prod_{p>z} p^{v_p(A)}$$

with $A = n(n - 1) \cdots (n - k + 1)$. Then

$$Q_z \ge \frac{n(n - 1) \cdots (n - k + 1)}{(k - 1)! \prod_{p \le z} p^{v_p(d_p)}} \ge \frac{(n - k + 1)^{k-\pi(z)}}{(k - 1)!},$$

where $\pi(z)$ denotes the number of primes $\le z$.

We write $z = 2\varepsilon k \log k$ in this case. By the prime number theorem we have

$$\pi(z) \le \frac{(1+\delta)2\varepsilon k \log k}{\log(2\varepsilon k \log k)}.$$

We get

$$\begin{aligned} Q_z &\ge \left(\frac{1}{2}n\right)^{k-(1+\delta)2\varepsilon k\log k/\log(2\varepsilon k\log k)} (\varepsilon n \log k)^{-23k/42} \\ (10) \quad &\ge \left(n^{19/42-(1+\delta)2\varepsilon\log k/\log(2\varepsilon k)}(\varepsilon\log k)^{-23/42}2^{(1+\delta)2\varepsilon\log k/\log(2\varepsilon k)-1}\right)^k, \end{aligned}$$

where we have used that $n-k+1 \ge \frac{1}{2}n$ and $(k-1)! \le k^k \le (tn)^{23k/42} < (\varepsilon n \log k)^{23k/42}$. By definition of Q_z we now just have to guarantee that the exponent of n in the right hand side of (10) is > 0, then the existence of a prime p with the required properties follows. The exponent of n is $19/42 - (1+\delta)2\varepsilon\log k/\log(2\varepsilon k) \ge 19/42 - 2\varepsilon(1+\delta)^2 > 0$, since $\log k/\log(2\varepsilon k) \le 1+\delta$, $\varepsilon = 1/5$ and $\delta = 1/16$, implying $Q_z > 1$. This completes the proof in the first case.

Now we turn to the second case and assume that $k < n^{11/21}$. In this case we can immediately improve the lower bound for Q_z in the arguments above to

$$(11) \qquad Q_z \ge \left(\frac{1}{2}n\right)^{k-(1+\delta)2\varepsilon k\log k/\log(2\varepsilon k)} n^{-11k/21} \ge n^{k/41}2^{-k} \ge n^{k/42},$$

since $10/21 - (1+\delta)2\varepsilon\log k/\log(2\varepsilon k\log k) \ge 10/21 - 2\varepsilon(1+\delta)^2 \ge 1/41$. Now let $p > z$ be a prime with $p|n(n-1)\cdots(n-k+1)$. Note that since $p \ge tk+s+1 > k$ it follows that p divides exactly one of $n, n-1, \ldots, n-k+1$. Assume that $p|n-\ell$ with $\ell \in \{0,\ldots,k-1\}$ and let $r > 0$ be such that $p^r \| n-\ell$. Moreover, observe that (with the notation of Lemma 5) we have $r = v_p(A)$. Since we are assuming that $F(x)$ has a factor of degree k it follows by Lemma 4 applied with this prime p that we must have

$$\frac{\log(tn+s)}{p^{v_p(A)}\log p} + \frac{1}{p-1} = \frac{\log(tn+s)}{p^r \log p} + \frac{1}{p-1} > \frac{1}{k}.$$

Since $p > (t+1)k \ge 3k$, we have $1/(p-1) \le 1/(3k)$ and therefore we deduce from the last formula that

$$p^{v_p(A)} < \frac{3k\log((t+1)n)}{2\log p} \le \frac{3\left(1+\frac{\log(t+1)}{\log n}\right)}{2\log((t+1)k)} k\log n < \frac{1}{43}k\log n;$$

this is true since $(129/2)(\log(t+1)+\log n) < \log n \log(t+1) + \log n \log k$. Therefore, we get

$$p < \frac{1}{43} k \log n \quad \text{and} \quad v_p(A) < \frac{\log k + \log\log n - \log 43}{\log p}$$

and hence

$$\sum_{p>z} v_p(A) \log p < \sum_{z<p\le \frac{1}{43} k \log n} (\log k + \log\log n - \log 43) \le \frac{1}{42} k \log n.$$

By comparing this upper bound for $\log Q_z$ with the lower bound from (11) we immediately end up with a contradiction. This proves the assertion in the second case.

3.2. **The case of positive indices, i.e. the case** (2)**.** Assume first that $n^{23/42} \le k \le \frac{n}{2}$. By Lemma 2 we get that there is a prime p of the form $p = (t+1)n + s - \ell$ with

$$\ell \le ((t+1)n+s)^{47/86} \le (3n \log n)^{47/86} < n^{23/42} \le k.$$

By the definition of the c'_j (see (5)) it follows that p divides $c'_{\ell+1}, \dots, c'_n$, and clearly it does not divide $c'_0 = 1$. Therefore $v_p(c'_j) \ge 1$ for $j \in \{\ell+1, \dots, n\}$ and $v_p(c'_0) = 0$. Since $p > tn + s$ and therefore $2p > 2tn + 2s \ge (t+1)n + s$, we see that $(t+1)n + s - \ell = p$ is the only multiple of p among the numbers $(t+1)n + s - j$ with $0 \le j < n$. Finally, from $c'_n = (-1)^n (tn+s+1) \cdots ((t+1)n+s)$ we get that $v_p(c'_n) = 1$. Therefore, the right-most edge of the Newton polygon of $f(x)$ has slope $< 1/(n-k) \le 1/k$. This is not possible by Lemma 1 and therefore there is no factor of degree k in the range.

Now we can assume that $k < n^{23/42}$. We write $z = 6\varepsilon k \log k$ in this case. By observing that $(t+1)n + s - k + 1 \ge tn + s \ge n$, we get from Lemma 5

$$Q_z((t+1)n+s, k) \ge n^{k-\pi(z)} n^{-23/42k} \ge \left(n^{19/42-(1+\delta)6\varepsilon \log k/\log(6\varepsilon k)}\right)^k > 1,$$

since $19/42 - 6\varepsilon(1+\delta)^2 > 0$. It follows that there is a prime $p > z \ge 2k$ dividing $(t+1)n + s - \ell$ with $0 \le \ell \le k-1$. For such a prime it follows that $p | c'_j$ for $j \in \{\ell+1, \dots, n\}$ and we define $m = m(p) \in \{1, \dots, n\}$ such that

$$\frac{v_p(c'_n) - v_p(c'_{n-m})}{m}$$

is the slope of the right most edge of the Newton polygon for $f(x)$ with respect to p. Then we have by Lemma 1 and (6) that

$$\begin{aligned}\frac{1}{k} \le \frac{v_p(c'_n) - v_p(c'_{n-m})}{m} &\le \frac{1}{m}\left[v_p\left(\frac{(tn+s+m)!}{(tn+s)!}\right) - v_p\left(\binom{n}{m}\right)\right] \\ &\le \frac{1}{m} v_p((tn+s+1)\cdots(tn+s+m)) \\ &\le \frac{1}{m}\sum_{j=1}^{\infty}\left(\left\lfloor\frac{tn+s+m}{p^j}\right\rfloor - \left\lfloor\frac{tn+s}{p^j}\right\rfloor\right) \\ &\le \frac{1}{m}\sum_{j=1}^{J}\left(\frac{m}{p^j}+1\right) \le \frac{1}{p-1} + \frac{J}{m} \le \frac{1}{2k} + \frac{J}{m},\end{aligned}$$

where

$$J := \left\lfloor \frac{\log((t+1)n+s)}{\log p} \right\rfloor.$$

It follows that $m \le 2kJ$ and thus

$$m \le \frac{2k\log((t+1)n+s)}{\log p} < \frac{4k\log n}{\log k} =: m_0. \tag{12}$$

So this inequality is true for all primes $p > z$ dividing $(t+1)n+s-\ell$ with $0 \le \ell \le k-1$.

Let $U := \{(t+1)n+s, \dots, (t+1)n+s-k+1\}\backslash\{b_q : q \le z\}$, where for all primes $q \le z$ we have removed those numbers $b_q \in \{(t+1)n+s, \dots, (t+1)n+s-k+1\}$ with $v_q(b_q)$ maximal. We mention, as we have already seen, that all elements of U are $\ge n$. Now let Ω be the set of all primes $q > z$ with $v_q(u) > 0$ from some $u \in U$ and $q^{v_q(u)} \le 2\varepsilon(m+k)\log k$ for all $u \in U$. Here we recall that $m = m(q)$ satisfies (12), since $q > z$ and divides some $u \in U$. Observe that such a q divides exactly one $u \in U$, since $q > z \ge k$. Thus we have

$$\begin{aligned}\log\left(\prod_{u\in U}\prod_{q\in\Omega} q^{v_q(u)}\right) &\le \log\left(\prod_{z<q\le 2\varepsilon(m_0+k)\log k} 2\varepsilon(m_0+k)\log k\right) \\ &\le \pi(2\varepsilon(m_0+k)\log k)\log(2\varepsilon(m_0+k)\log k) \\ &\le (1+\delta)2\varepsilon(m_0+k)\log k \\ &\le 12(1+\delta)\varepsilon k\log n,\end{aligned}$$

by using (12). It follows that

$$\log\left(\prod_{u\in U}\prod_{q\le z} q^{v_q(u)}\prod_{q\in\Omega} q^{v_q(u)}\right)\le k\log k+12(1+\delta)\varepsilon k\log n\le \frac{2}{3}k\log n,$$

since $k<n^{23/42}$. On the other hand we have

$$\log\left(\prod_{u\in U} u\right)\ge (k-\pi(z))\log n\ge \left(1-\frac{(1+\delta)6\varepsilon\log k}{\log(6\varepsilon k)}\right)k\log n>\frac{2}{3}k\log n,$$

since $\log k/\log(6\varepsilon k)\le 1+\delta$. By putting the last two statements together we conclude that there is a prime $q>z$ that divides some element $u\in U$ with the additional property that $q^{v_q(u)}>2\varepsilon(m+k)\log k$. Let $u=(t+1)n+s-\ell$ with $0\le\ell\le k-1$ and let r be defined by $q^r>2\varepsilon(m+k)\log k\ge q^{r-1}$ and such that q^r divides u. Again, by (5), it follows that this prime divides c'_j for $j\in\{\ell+1,\dots,n\}$. If q^r divides $tn+s+j$ for some $1\le j\le m$, then it also divides $(t+1)(tn+s+j)-t((t+1)n+s-\ell)=s+tj+t\ell+j$ and therefore $q^r\le s+(t+1)m+t(k-1)\le 2\varepsilon(m+k)\log k$, which is a contradiction. Therefore, q^r does not divide $tn+s+j$ for all $1\le j\le m$. Hence, we now have

$$\begin{aligned}\frac{1}{k}\le\frac{v_q(c'_n)-v_q(c'_{n-m})}{m}&\le\frac{1}{m}v_q((tn+s+1)\cdots(tn+s+m))\\ &\le\frac{1}{m}\sum_{j=1}^{r-1}\left(\frac{m}{q^j}+1\right)\le\frac{1}{q-1}+\frac{r-1}{m}<\frac{1}{3k}+\frac{r-1}{m},\end{aligned}$$

since $q>z\ge 3k$. For $r\ge 2$ we have

$$2(2\varepsilon(m+k)\log k)\ge 2q^{r-1}>2(6\varepsilon k\log k)^{r-1}\ge(6\varepsilon k\log k)^{r-1}+4\varepsilon k\log k$$

and therefore

$$4\varepsilon m\log k>(6\varepsilon k\log k)^{r-1}\ge(r-1)6\varepsilon k\log k.$$

Finally we conclude

$$m>\frac{3}{2}k(r-1)$$

and

$$\frac{1}{k}\le\frac{v_q(c'_n)-v_q(c'_{n-m})}{m}<\frac{1}{3k}+\frac{2}{3k}=\frac{1}{k},$$

which is a contradiction. This completes the proof of Theorem 1.

4. Proof of Theorem 3

We take $\varepsilon = 1/7$ and $\delta = 1/40$ in the proof of Theorem 3. Further, let η_3 be sufficiently large. Assume that $F(x)$ has a factor of degree $k \leq \frac{1}{2}n$ such that $t < \varepsilon \log n, s < \varepsilon k \log n$ with $n > \eta_3$.

We can start as in the proof of Theorem 1 and conclude that, actually, we may assume that $k \leq (tn)^{23/42}$. Moreover, we can follow the arguments given there afterwards, but for k in the range $k \geq (n/2)^{2/3}$. Namely, by applying Lemma 5 and the prime number theorem we get that for $z = 2\varepsilon k \log n$ we have

$$\begin{aligned} Q_z &\geq \left(\frac{1}{2}n\right)^{k-\pi(z)} (tn)^{-23k/42} \\ &\geq \left(n^{\frac{19}{42}-\frac{(1+\delta)2\varepsilon \log n}{\log(2\varepsilon(n/2)^{2/3}\log n)}} (\varepsilon \log n)^{-\frac{23}{42}} 2^{-1+\frac{(1+\delta)2\varepsilon\log n}{\log(2\varepsilon(n/2)^{2/3}\log n)}}\right)^k, \end{aligned}$$

which shows, since $\varepsilon = 1/7$ and $\log n / \log(2\varepsilon(n/2)^{2/3}\log n) \leq 3/2$, that there is a prime $p > 2\varepsilon k \log n > \varepsilon n^{2/3}$ and $p \geq tk + s + 1$ that divides $n(n-1)\cdots(n-k+1)$. For this prime we have $tn + s < \varepsilon n \log n + \varepsilon k \log n \leq 2\varepsilon n \log n$ and

$$\frac{\log(tn+s)}{p \log p} + \frac{1}{p-1} \leq \frac{\log(2\varepsilon n \log n)}{2\varepsilon k \log n \log\left(\varepsilon n^{2/3}\right)} + \frac{1}{\varepsilon k \log n} \leq \frac{1}{k}$$

and therefore it follows by Lemma 4 that $F(x)$ cannot have a factor of degree k.

Let η_6, η_7 and η_8 be the constants appearing in Lemma 3. We are left with $k \leq (n/2)^{2/3}$, i.e. with $2k^{3/2} \leq n$. Since we are assuming that $\log n \leq (\log k)^{3/2(1+\varepsilon)^{-1}} \leq (\log k)^{3/2(1-\varepsilon/2)}$ it follows that

$$\log n \leq \eta_6 \frac{(\log k)^{3/2}}{\log \log k}$$

and therefore that

$$k^{3/2} \leq \frac{n}{2} \leq u := n - k \leq n \leq k^{\eta_6 (\log k)^{1/2}/\log\log k}. \tag{13}$$

Hence, by the same lemma we get that the largest prime factor $p = P(u,k)$ of $n(n-1)\cdots(n-k+1) = (u+1)(u+2)\cdots(u+k)$ satisfies

$$p \geq \eta_8 k^{1+\eta_7 \Lambda(k,u)}.$$

We observe

$$\Lambda(k,u) \geq \left(\frac{\log k}{\log n}\right)^2 \geq \eta_6^{-2}\left(\frac{\log\log k}{(\log k)^{1/2}}\right)^2 \geq \eta_9 \frac{(\log\log n)^2}{\log k}$$

where we have used the last inequality in (13) again. Thus, $p \geq \eta_{10} k(\log n)^2 > k\log n$ and $p > 2\varepsilon k \log n > tk+s$. But since

$$\frac{\log(tn+s)}{p\log p} + \frac{1}{p-1} \leq \frac{\log(2\varepsilon n\log n)}{k\log n\log(\log n)} + \frac{1}{p-1} \leq \frac{1}{k}$$

it follows again by Lemma 4 that $F(x)$ cannot have a factor of degree k. This is a contradiction.

This completes the proof of Theorem 3.

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