DIOPHANTINE INEQUALITIES INVOLVING SEVERAL POWER SUMS

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Abstract. Let $\mathcal{E}_A$ denote the ring of power sums, i.e. complex functions of the form

$$G_n = a_1\alpha_1^n + a_2\alpha_2^n + \ldots + a_t\alpha_t^n,$$

for some $a_i \in \mathbb{C}$ and $\alpha_i \in A$, where $A \subseteq \mathbb{C}$ is a multiplicative semigroup. Moreover, let $F(n,y) \in \mathcal{E}_A[y]$. We consider Diophantine inequalities of the form

$$\left|F(n,y)\right| < \alpha^{n(d-1-\epsilon)},$$

where $\alpha > 1$ is a quantity depending on the dominant roots of the power sums appearing as coefficients in $F(n,y)$, and show that all its solutions $(n,y) \in \mathbb{N} \times \mathbb{Z}$ have y parametrized by some power sums from a finite set.

This is a continuation of the work of Corvaja and Zannier [4, 5, 6] and of the authors [10, 18] on such problems.

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1. Introduction

Let $\mathcal{E}_A$ denote the ring of power sums, i.e. complex functions on $\mathbb{N}$ of the form

$$G_n = a_1\alpha_1^n + a_2\alpha_2^n + \ldots + a_t\alpha_t^n,$$

for some $a_i \in \mathbb{C}$ and $\alpha_i \in A$, where $A \subseteq \mathbb{C}$ is a multiplicative semigroup. If $K \subseteq \mathbb{C}$ is a field we define $K\mathcal{E}_A$ by the same formulas, but allowing $a_i \in K$. The a_i are called the coefficients of G_n and the α_i are called the roots.

It is well known that such a power sum G_n satisfies a certain linear recurring equation (see [10]). Namely, set

$$\prod_{i=1}^{t}(X-\alpha_i) = X^t - A_1X^{t-1} - \ldots - A_t \qquad (1)$$

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for $A_1, A_2, \dots, A_t \in \mathbb{C}$. Then the sequence (G_n) satisfies the t-th order linear recurring relation

$$G_n = A_1 G_{n-1} + \dots + A_k G_{n-k} \text{ for } n = k, k+1, \dots,$$

with initial values $G_0, G_1, \dots, G_{k-1} \in K$. We remark that the general solution of such a linear recurring relation is a "generalized" power sum being of the same form as above, but allowing the a_i to be polynomials in n and the $c_i \in \mathbb{C}$. Thus, we are considering linear recurring sequences (G_n), where all roots of the corresponding characteristic polynomial are simple.

Below, A will be usually $\mathbb{Z}$; in that case we define $\mathcal{E}_{\mathbb{Z}}^{+}$ the subring formed by those power sums having only positive roots, i.e. the roots belong to the semigroup $\mathbb{N}$. Working in this domain causes no loss of generality: this assumption may be achieved by writing $n = 2m + r$ and considering the cases $r = 0, 1$ separately.

The power sum G_n is called nondegenerate, if no quotient α_i/α_j for $1 \le i < j \le t$ is equal to a root of unity. Observe that restricting to nondegenerate recurring sequences causes no substantial loss of generality.

In the present paper we are dealing with Diophantine problems, where power sums are involved. E.g. many Diophantine equations involving linear recurring sequences were studied earlier, for instance in the special case

$$G_n = Ex^q, \quad E \in \mathbb{Z}\backslash\{0\}.$$

A survey about this equation can be found in [14, 15] and in more general form in [8, 11, 10].

Also Diophantine inequalities were studied previously, we just mention a result due to Shorey and Stewart [19], who proved that for any fixed $\epsilon > 0$ the inequality

$$|Ex^q - G_n| > |\alpha_1|^{n(1-\epsilon)},$$

where $G_n \in \mathcal{E}_{\overline{\mathbb{Q}}}$ (where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$) is nondegenerate, holds for all nonzero integers E, x, for $n > 0$, and for every nonzero integer $q > q_0(G_n, P)$, where P is the greatest prime factor of E, assuming that $Ex^q \neq c_1\alpha_q^n$ and that in G_n there is a root with largest absolute value. This result was obtained by the application of lower bounds for linear forms in logarithms of algebraic numbers due to Baker [1].

In 1998, a new development was started by Corvaja and Zannier [4]. They considered power sums defined by

$$G_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_t\alpha_t^n,$$

where $t \ge 2, c_1, c_2, \dots, c_t$ are non-zero rational numbers, $\alpha_1 > \alpha_2 > \cdots > \alpha_t > 0$ are integers. They used Schmidt's Subspace Theorem [16], [17] to show that for fixed $\epsilon > 0$ and every integer $q \ge 2$ there exist power sums $H_n^{(1)},\dots,H_n^{(s)} \in \overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$ such that all solutions $(n,y)\in\mathbb{N}\times\mathbb{Z}$ of the Diophantine inequality

$$|y^q - G_n| \ll |G_n|^{1-\frac{1}{d}-\epsilon}$$

apart from finitely many, satisfy $y = H_n^{(i)}$ for a certain $i = 1,\dots,s$. As a consequence, for every $q \geq 2$ the equation $G_n = x^q$ has only finitely many solutions, if we suppose that α_1,α_2 are coprime.

From this point on two developments started. On the one hand more general Diophantine equations, where power sums (or linear recurring sequences) are involved were studied (e.g. cf. [5, 6]). Also quantitative aspects were handled in view of the existence of quantitative version of the Subspace Theorem (e.g. due to Evertse [7]) (cf. [11, 8]). On the other hand new results for Diophantine inequalities were obtained (e.g. [2, 9]).

Recently, the second author studied lower bounds for the quantity $|F(G_n, y)|$, where $F(x,y)\in\overline{\mathbb{Q}}[x,y]$ is absolutely irreducible, monic and of degree $d\geq 2$ in y. He proved that for $G_n\in\overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}$ and for fixed $\epsilon>0$ there exists a finite set of power sums $H_n^{(1)},\dots,H_n^{(s)}\in\mathcal{E}_{\mathbb{Z}}^{+}$ such that every solution $(n,y)\in\mathbb{N}\times\mathbb{Z}$, apart from finitely many, of the Diophantine inequality

$$\Big|F(G_n,y)\Big| < \left|\frac{\partial F}{\partial y}(G_n,y)\right| \cdot |G_n|^{-\epsilon}$$

satisfies $y = H_n^{(i)}$ for a certain $i = 1,\dots,s$.

Very recently, the authors considered Diophantine equations, where more than one power sum is involved [10]. The aim of the present paper is to generalize this result to Diophantine inequalities, where also more than one power sum G_n is involved.

First we need some notation. Let $d\geq 2$ be an integer and let $G_n^{(1)},\dots,G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$, i.e. we have

$$\begin{aligned} G_n^{(1)} &= a_1^{(1)}{\alpha_1^{(1)}}^n + a_2^{(1)}{\alpha_2^{(1)}}^n + \dots + a_{t(1)}^{(1)}{\alpha_{t(1)}^{(1)}}^n, \\ &\vdots \\ G_n^{(d)} &= a_1^{(d)}{\alpha_1^{(d)}}^n + a_2^{(d)}{\alpha_2^{(d)}}^n + \dots + a_{t(d)}^{(d)}{\alpha_{t(d)}^{(d)}}^n, \end{aligned}$$

where $a_i^{(j)}$ are algebraic and $\alpha_i^{(j)}$ are positive integers such that $\alpha_1^{(j)} > \alpha_2^{(j)} > \dots > \alpha_{t(j)}^{(j)}$ for all $i = 1,\dots,t^{(j)}$ and $j = 1,\dots,d$.

We are studying lower bounds for the quantity

$$\big|F(n,y)\big|,$$

where $F \in \mathcal{E}_{\mathbb{Z}}^{+}[y]$. This is equivalent to study bounds for

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right|.$$

Now, we set (for a positive real determination of the roots)

$$\alpha := \max\left\{ \alpha_1^{(d)\frac{1}{d}}, \left(\alpha_1^{(d-1)}\right)^{\frac{1}{d-1}}, \left(\alpha_1^{(d-2)}\right)^{\frac{1}{d-2}}, \ldots, \alpha_1^{(1)} \right\} = \max_{i=1,\ldots,d} \left(\alpha_1^{(i)}\right)^{\frac{1}{i}}.$$

Moreover, let

$$y = \alpha^n z.$$

Then consider

$$\frac{1}{\alpha^{dn}}\Big((\alpha^n z)^d + G_n^{(1)}(\alpha^n z)^{d-1} + \ldots + G_n^{(d)}\Big). \tag{2}$$

This is a polynomial in z with coefficients in $\overline{\mathbb{Q}}\mathcal{E}_A$, where A is the multiplicative group generated by

$$\alpha \text{ and the roots of } G_n^{(1)}, \ldots, G_n^{(d)}.$$

Observe that all the roots which appear in the coefficients are ≤ 1, because of our construction and that the coefficient of z^d is 1 (cf. Lemma 2 in [10]). Let $\gamma_1, \ldots, \gamma_r$ denote the different roots of these power sums, which are strictly less than 1. Identifying the expressions γ_i^n in (2) by new variables x_i, we get a polynomial (linear in $x_1, \ldots, x_r$) $g(x_1, \ldots, x_r, z) \in \overline{\mathbb{Q}}[x_1, \ldots, x_r, z]$ such that

$$g(\gamma_1^n, \ldots, \gamma_r^n, z) = \frac{1}{\alpha^{dn}}\Big((\alpha^n z)^d + G_n^{(1)}(\alpha^n z)^{d-1} + \ldots + G_n^{(d)}\Big). \tag{3}$$

This polynomial is some kind of normal form for our problem. We denote by $D(G_n^{(1)}, \ldots, G_n^{(d)})$ the discriminant of g with respect to z evaluated at $(0, \ldots, 0)$, i.e.

$$D(G_n^{(1)}, \ldots, G_n^{(d)}) = \mathrm{disc}_z(g)(0, \ldots, 0).$$

The main result of the authors in [10] was: Let $d \geq 2$ and let $G_n^{(1)}, \ldots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Assume that

$$D(G_n^{(1)}, \ldots, G_n^{(d)}) \neq 0. \tag{4}$$

Then there exist finitely many recurrences $H_n^{(1)}, \ldots, H_n^{(s)}$ with algebraic coefficients and algebraic roots, arithmetic progressions $\mathcal{P}_1, \ldots, \mathcal{P}_s$ such that for the set S of solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the equation

$$y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d-1)} y + G_n^{(d)} = 0$$

we have

$$S = \bigcup_{i=1}^{s} \{(n, H_n^{(i)}) : n \in \mathcal{P}_i\} \cup M,$$

where M is a finite set. In fact, they also allowed the coefficient of y^d in the equation above to be a power sum $G_n^{(0)}$, which leads to another possible "trivial" infinite family of solutions, but for simplicity we will always assume the above situation in this paper.

2. Results

From the discussion above we are searching for an exponential lower bound for the quantity

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right| \tag{5}$$

(in terms of α to some power k). In the following we divide $\mathbb{N} \times \mathbb{Z}$ in two regions depending on whether $|y|$ is above or below the curve $c\alpha^n$, where

$$c := \sum_{i=1}^{d} \left| a_1^{(i)} \right| + 1. \tag{6}$$

In fact the behavior of (5) depends on the region where we are looking for solutions. First it is easy to get the following statement.

Proposition 1. *Let $d \geq 2$, $G_n^{(1)}, \ldots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Let*

$$\mathcal{U} = \{(n, y) \in N \times \mathbb{Z} \, : \, |y| > c\alpha^n\}.$$

Then the Diophantine inequality

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right| \leq \alpha^{nd},$$

has only finitely many solutions $(n, y) \in \mathcal{U}$.

Remark 1. *We mention that the result is not true if the exponent on the right hand side increases, i.e. the exponent is best possible. Let us consider, for a given $\epsilon > 0$, the Diophantine inequality*

$$\left| y^3 - a^n \right| < \alpha^{3n(1+\epsilon)},$$

where $\alpha = a^{\frac{1}{3}}, a > 1$. We have $c = 2$. Putting

$$y_n = \left\lfloor 2a^{\frac{n}{3}} + 1 \right\rfloor,$$

we have

$$\left| y_n^3 - a^n \right| < 8a^n < a^{n(1+\epsilon)}$$

for large n.

In the lower part of the plane, i.e. in $\mathcal{L} = \{(n, y) \in \mathbb{N} \times \mathbb{Z} \, : \, |y| \leq c\alpha^n\}$, we cannot expect to get just finitely many solutions, in view of the results (cf. [10]) on the equation

$$y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} = 0,$$

which can have a functional solution.

Our main result is the following characterization of the solutions:

Theorem 1. *Let $d \geq 2$, $G_n^{(1)}, \ldots, G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$. Assume that*

$$D(G_n^{(1)}, \ldots, G_n^{(d)}) \neq 0. \tag{7}$$

Finally, let $\epsilon > 0$. Then there exist finitely many recurrences $H_n^{(1)}, \ldots, H_n^{(s)} \in \overline{\mathbb{Q}}\mathcal{E}_{\overline{\mathbb{Q}}}$ such that all the solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the Diophantine inequality

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right| < \alpha^{n(d-1-\epsilon)}, \tag{8}$$

except finitely many, have $y = H_n^{(i)}$ for some $i = 1, \ldots, s$. Moreover, the set of natural numbers n such that (n, y) is a solution of the inequality is the union of a finite set and a finite number of arithmetic progressions.

Remark 2. *Observe that for $G_n^{(1)} = \ldots = G_n^{(d-1)} = 0$ and $G_n^{(d)} = -G_n \in \mathcal{E}_{\mathbb{Z}}^{+}$ we get the conclusion for the inequality*

$$\left| y^d - G_n \right| < \alpha^{n(d-1-\epsilon)} = \alpha_1^{n\left(1-\frac{1}{d}-\epsilon'\right)},$$

where α_1 is the dominant root of G_n, i.e. the result of Corvaja and Zannier [4]*. This also shows (cf. Remark 2, page 321 in* [4]*) that the exponent $d - 1 - \epsilon$ in Theorem 1 is best possible.*

Remark 3. *Consider*

$$G_n^{(1)} = p_1(G_n), G_n^{(2)} = p_2(G_n), \ldots, G_n^{(d)} = p_d(G_n),$$

where $p_j(x) \in \overline{\mathbb{Q}}[x]$ are polynomials of degree i_j for $j = 1, \ldots, d$ and where $G_n \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}$. Let α_1 denote the dominant root of G_n. Then,

$$\alpha = \max\left\{ \alpha_1^{\frac{i_d}{d}}, \alpha_1^{\frac{i_{d-1}}{d-1}}, \ldots, \alpha_1^{i_1} \right\} \geq \alpha_1^{\frac{1}{d}}.$$

We get our conclusion for the inequality

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right| < \alpha^{n(d-1-\epsilon)}.$$

In comparison the result in [18] *implies the same conclusion only for*

$$\left| y^d + G_n^{(1)} y^{d-1} + \ldots + G_n^{(d)} \right| < \alpha_1^{n\left(1-\frac{1}{d}-\epsilon\right)}.$$

This means that although the exponent in the result in [18] *is best possible, the upper bound is not (and this is clear, because it does not take care of the special structure of the polynomials $p_1, \ldots, p_d$). Therefore, Theorem 1 above is an improvement of this result. Observe that the upper bounds are equal only in the case*

$$G_n^{(1)} = a_1, \ldots, G_n^{(d-1)} = a_d, \quad a_1, \ldots, a_d \in \overline{\mathbb{Q}}$$

which is more or less the case from Remark 2 above.

Remark 4. *As we use the Implicit Function Theorem as our driving tool, it is clear that condition (4) is equivalent to*

$$(4) \iff g(0,\dots,0,z) \text{ has only simple roots } \iff$$
$$\iff \frac{\partial g}{\partial z}(0,\dots,0,z_i) \neq 0, \ i = 1,\dots,d,$$

where z_i are the roots of $g(0,\dots,0,z)$. Note that a similar condition already appeared in the main result in [5].

Remark 5. *Observe that it is easy to verify whether the inequality has infinitely many solutions or not (cf.* [18, Remark 5.1] *and* [10, Remark 2]*).*

As in [18] we get as a simple application of our main theorem the following result, which generalizes the result in [10]:

Corollary 1. *Let $d \geq 2$, $G_n^{(1)},\dots,G_n^{(d)} \in \overline{\mathbb{Q}}\mathcal{E}_{\mathbb{Z}}^{+}$ not all constant. Moreover, let $f(x) \in \mathbb{Z}[x]$ be a non constant polynomial. Assume that*

$$D(G_n^{(1)},\dots,G_n^{(d)}) \neq 0.$$

Then the Diophantine equation

$$y^d + G_n^{(1)} y^{d-1} + \dots + G_n^{(d)} = f(n),$$

has only finitely many solutions $(n,y) \in \mathbb{N} \times \mathbb{Z}$.

Moreover, we get the following result, which looks quite interesting: Let $N_y(n)$ denote the number of solutions of the Diophantine inequality

$$\left| y^d + G_n^{(1)} y^{d-1} + \dots + G_n^{(d)} \right| < \alpha^{n(d-1-\epsilon)},$$

for fixed n. Obviously, we have $N_y(n) < \infty$ for all $n \in \mathbb{N}$, because the left hand side of the above inequality goes to ∞ as $|y|$ becomes large. Surprisingly, the above result implies that there exists a uniform upper bound not depending on n.

Corollary 2.

$$N_y(n) \leq C,$$

where C is a constant which does not depend on n.

Remark 6. *Moreover, observe that the constant C is effectively computable in view of the existence of quantitative versions of the Subspace Theorem, e.g. due to Evertse* [7].

3. Auxiliary results

The proof of our theorem depends on a technical result due to Corvaja and Zannier [6, Theorem 4], which is derived as a consequence from the famous Subspace Theorem of W. Schmidt (cf. [16, 17]).

Let K be an algebraic number field. Denote its collection of places by M_K and let S be a finite set of absolute values of K containing the archimedean ones. For every place v of K we note by $|\cdot|_v$ a continuation of it to $\overline{\mathbb{Q}}$ and normalize it "with respect to K": according to this normalization, for $x \in K^*$ the absolute logarithmic Weil height is

$$h(x) = \sum_{v \in M_K} \log \max\{1, |x|_v\}$$

and the *Product formula*

$$\prod_{v \in M_K} |x|_v = 1 \tag{9}$$

holds. We note that these conditions uniquely determine our normalizations.

We also define the S-height of a non zero element $x \in K^*$ to be

$$h_S(x) = \sum_{v \notin S} \log \max\{1, |x|_v\}.$$

For S-integers this height vanishes, so it measures "how far" x is from being an S-integer.

For a vector $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in K^{n+1} \backslash \{\mathbf{0}\}, (n \geq 1)$, we define $h(\mathbf{x})$ as the usual projective logarithmic height. Also, we denote by $\hat{h}(\mathbf{x})$, the sum of the $h(x_i), 0 \leq i \leq n$. Moreover, we put, for an absolute value v,

$$\|\mathbf{x}\|_v := \max_{1 \leq i \leq n} \{|x_i|_v\}.$$

We use $\mathcal{H}, \mathcal{H}_S$ and $\hat{\mathcal{H}}$ for the exponential heights.

Theorem 2 (Corvaja and Zannier). *Let K be a number field, S a finite set of absolute values of K containing all the archimedean ones, v be an absolute value from S, ϵ be a positive real number, $N \geq 0$ an integer. Finally, let $c_0, \ldots, c_N \in \overline{K}^*$. For $\delta > (N+2)\epsilon$, there are only finitely many $(N+1)$-tuples $\mathbf{w} = (w_0, \ldots, w_N) \in (K^*)^{N+1}$ such that the inequalities*

(i) $\quad h_S(w_i) + h_S(w_i^{-1}) \leq \epsilon h(w_i), \quad$ *for* $i = 1, \ldots, N$

(ii) $\quad |c_0 w_0 + c_1 w_1 + \ldots + c_N w_N|_v < \left(\mathcal{H}(w_0) \mathcal{H}_S(w_0)^{N+1}\right)^{-1} \hat{\mathcal{H}}(\mathbf{w})^{-\delta}$

hold and no subsum of the $c_i w_i$ involving $c_0 w_0$ vanishes.

Our second tool is the Implicit Function Theorem. The basic form of the Implicit Function Theorem is the assertion that a function in n variables, of sufficient smoothness, satisfying an appropriate nondegeneracy condition, can be used to define one of the variables as a function of the other $n-1$ variables. Here we will consider the implicit function theorem in the real analytic category (see [12], page 35 and [13]). We will use the following notation: a *multiindex* α is an element of $\mathbb{N}^m$. Set

$$|\alpha| = |\alpha_1 + \ldots + \alpha_m|.$$

We will write 0 to mean the multiindex $(0, \ldots, 0)$.

Theorem 3 (Implicit Function Theorem). *Suppose the power series*

$$F(x_1,\dots,x_r,y)=\sum_{|\alpha|\geq 0,k\geq 0} a_{\alpha,k}x_1^{\alpha_1}\cdots x_r^{\alpha_r}y^k$$

is absolutely convergent for $|x_1|+\ldots+|x_r|\leq R_1, |y|\leq R_2$. *If*

$$a_{0,0}=0 \text{ and } a_{0,1}\neq 0$$

then there exist $r_0>0$ *and a power series*

$$f(x_1,\dots,x_r)=\sum_{|\alpha|>0} c_\alpha x_1^{\alpha_1}\cdots x_r^{\alpha_r} \tag{10}$$

such that (10) is absolutely convergent for $|x_1|+\ldots+|x_r|\leq r_0$ *and*

$$F(x_1,\dots,x_r,f(x_1,\dots,x_r))=0.$$

Moreover: if the coefficients of F *are algebraic, then the coefficients of* f *are also algebraic.*

We mention once again the lemma from [10] that assures that the construction leading to the polynomial g appearing in (3) has the properties we have claimed.

Lemma 1. *Let* $G_n^{(1)},\dots,G_n^{(d)}$ *and* α *be as in the Introduction. Then the dominant root in*

$$(\alpha^n z)^d+G_n^{(1)}(\alpha^n z)^{d-1}+\ldots+G_n^{(d)}$$

is α^d *and it appears as coefficient of* z^d *and as coefficient of* z^{d-k} *for all indices* k *with*

$$\alpha=\alpha_1^{(k)\frac{1}{k}},$$

i.e. for all indices where the maximum in the definition of α *appears.*

This is Lemma 2 in [10]. The additional statement can be found as a remark at the end of the proof in [10].

4. Proof of Proposition 1

We will prove the statement by contradiction. Let us assume that the inequality

$$\left|y^d+G_n^{(1)}y^{d-1}+\ldots+G_n^{(d)}\right|\leq\alpha^{nd}$$

has infinitely many solutions $(n,y_n)\in\mathcal{U}$ with $n\in\Sigma$, where Σ is a sequence of positive integers. By setting

$$y_n=\alpha^n z_n$$

and by dividing through by α^{nd} we get

$$\left|g(\gamma_1^n,\dots,\gamma_r^n,z_n)\right|\leq 1,$$

with the polynomial g defined in (3).

First, we show by another indirect argument that the sequence $(z_n)_{n\in\Sigma}$, for the corresponding sequence $(n, z_n) \in \mathbb{N}\times K$ with $n \in \Sigma$, is bounded by c as defined in (6). Suppose that $|z_n| > c > 1$ for some $n \in \Sigma$. We can write (see Lemma 1)

$$g(x_1,\dots,x_r,z) = z^d + p_1(x_1,\dots,x_r)z^{d-1} + \dots + p_d(x_1,\dots,x_r)$$

with polynomials $p_i(x_1,\dots,x_r)$. Dividing by $|z_n^{d-1}|$, we get

$$\left|z_n + p_1(\gamma_1^n,\dots,\gamma_r^n) + \dots + p_d(\gamma_1^n,\dots,\gamma_r^n)z_n^{-d+1}\right| \le \left|z_n^{-d+1}\right| \le 1-\epsilon,$$

for some $\epsilon > 0$. By Lemma 1 we have

$$\lim_{n\to\infty} p_i(\gamma_1^n,\dots,\gamma_r^n) = a_1^{(i)}$$

for all $i = 1,\dots,d$ and thus by reminding the definition of c we get

$$\left|p_1(\gamma_1^n,\dots,\gamma_r^n) + \dots + p_d(\gamma_1^n,\dots,\gamma_r^n)z_n^{-d+1}\right| \le c-1+\epsilon,$$

which holds if n is large enough. Consequently,

$$\left|z_n\right| \le \left|p_1(\gamma_1^n,\dots,\gamma_r^n) + \dots + p_d(\gamma_1^n,\dots,\gamma_r^n)z_n^{-d+1}\right| + \left|z_n^{-d+1}\right| \le c,$$

which is a contradiction, that proves that the sequence $(z_n)_{n\in\Sigma}$ is bounded.

We got that $|z_n| \le c$, which in turn implies

$$|y_n| = \alpha^n|z_n| \le c\alpha^n.$$

This means that our solutions do not lie in $\mathcal{U}$, which is a contradiction proving that there are at most finitely many n for which a solution $(n,y) \in \mathcal{U}$ exists. Since the inequality under consideration has clearly finitely many solutions y for any fixed n, we can conclude that it has only finitely many solutions $(n,y) \in \mathcal{U}$. □

5. Proof of Theorem 1

In the sequel $C_1, C_2, \dots$ will denote positive numbers depending only on the coefficients and roots of $G_n^{(1)},\dots,G_n^{(d)}$.

First we consider solutions of the form $(n,0), n \in \mathbb{N}$. In this case the inequality reduces to

$$\left|G_n^{(d)}\right| < \alpha^{n(d-1-\epsilon)}.$$

But in this case we trivially have finitely many solutions, if the dominant root of $G_n^{(d)}$ is $> \alpha^{d-1-\epsilon}$. In the case that $\alpha_1^{(d)} < \alpha^{d-1-\epsilon}$ the inequality is satisfied for all n large enough, i.e. for all n with finitely many exceptions. The last case is that

$$\alpha_1^{(d)} = \alpha^{d-1-\epsilon}.$$

Clearly, if $|a_1^{(d)}| > 1$ we have only finitely many solutions and if $|a_1^{(d)}| < 1$ we have solutions for all n large. If $|a_1^{(d)}| = 1$ we look at

$$G_n^{(d)} - a_1^{(d)}\alpha_1^{(d)^n}.$$

If n is large enough this is either always positive or negative and less than $a_1^{(d)}\alpha_1^{(d)^n}$. Therefore, depending on whether the sign of $a_1^{(d)}$ and $a_2^{(d)}$ are equal or not, we get, as before, that we have only finitely many solutions or solutions for all n with finitely many exceptions. Consequently, we can restrict ourselves to solutions of the form $(n,y) \in \mathbb{N} \times \mathbb{Z}$ with $y \neq 0$. Moreover, by Proposition 1, we can assume that $|y| \leq c\alpha^n$, i.e. that $(n,y) \in \mathcal{L}$.

Thus, in the following, we consider only solutions $(n,z) \in \mathbb{N} \times K$ of

$$\left|g(\gamma_1^n, \dots, \gamma_r^n, z)\right| < \alpha^{-n(1+\epsilon)} \tag{11}$$

with the polynomial g defined in (3) and with $0 < |z| \leq c$ as defined in (6), where K is the number field generated by the coefficients of $G_n^{(1)}, \dots, G_n^{(d)}$, and by α.

Now since $D(G_n^{(1)}, \dots, G_n^{(d)}) \neq 0$, by the Implicit Function Theorem 3 we can conclude that

$$g(x_1, \dots, x_r, z) = \left(z - \sum_{|i|\geq 0} a_{i,1} x_1^{i_1} \cdots x_r^{i_r}\right) \cdot \ldots \cdot \left(z - \sum_{|i|\geq 0} a_{i,d} x_1^{i_1} \cdots x_r^{i_r}\right) \tag{12}$$

with $a_{i,j} \in \overline{\mathbb{Q}}$, where $z_i := a_{0,i}, i = 1, \dots, d$ satisfy $g(0, \dots, 0, z_i) = 0$. The series

$$\varphi_1(x_1, \dots, x_r) = \sum_{|i|\geq 0} a_{i,1} x_1^{i_1} \cdots x_r^{i_r}, \ \dots, \ \varphi_d(x_1, \dots, x_r) = \sum_{|i|\geq 0} a_{i,d} x_1^{i_1} \cdots x_r^{i_r}$$

converge around the point $(x_1, \dots, x_r) = (0, \dots, 0)$. Observe that for n large enough each solution (n,z) of (11) gives rise to a point $(\gamma_1^n, \dots, \gamma_r^n)$ lying in the region of convergency of φ_j for some $j = 1, \dots, d$, i.e. we get that

$$z - \varphi_j(\gamma_1^n, \dots, \gamma_r^n) = z - \sum_{|i|\geq 0} a_{i,j} \gamma_1^{i_1 n} \cdots \gamma_r^{i_r n}$$

becomes arbitrarily small for some $j = 1, \dots, d$ and if n is large enough. We remark that here we need that $(n,y) \in \mathcal{L}$, which implies that the sequence of z is bounded.

From the assumption $D(G_n^{(1)}, \dots, G_n^{(d)}) \neq 0$ it follows that the z_i are distinct, because they are the roots of $g(0, \dots, 0, z) = 0$. This means that g is regular in the sense of the Definition in [18]. We consider now the sets

$$M_j = \left\{(n,z) \, : \, |z - \varphi_j| = \min_{i=1,\dots,d} \{|z - \varphi_i|\}\right\}, \ j = 1, \dots, d.$$

Here we have used the notation φ_i short for $\varphi_i(\gamma_1^n, \dots, \gamma_r^n)$. We will go on using this notation below. Without loss of generality we restrict ourselves to the case that $(n,z) \in M_1$ and show the conclusion of the theorem only in this case.

By considering $(n,z)\in M_1$ we get from (12) that

$$\left|g(\gamma_1^n,\ldots,\gamma_r^n,z)\right| = |z-\varphi_1|\,|z-\varphi_2|\,\cdots\,|z-\varphi_d|.$$

On the right hand side we have one "small" factor and all other factors are "big". We now want to calculate lower bounds for the quantities $|z-\varphi_i|$, $i=2,\ldots,d$.

We calculate the contribution of the "big" terms: for every $i\geq 2$ we have

$$|z-\varphi_i|\geq \frac{1}{2}|\varphi_i-\varphi_1|\geq \frac{1}{3}|z_i-z_1|,$$

for n large enough (recall that we are considering only those (n,z) which lie in M_1, i.e. we have $|z-\varphi_1|\leq|z-\varphi_i|$). Therefore, we conclude

$$|z-\varphi_2|\cdots|z-\varphi_d|\geq\left(\frac{1}{3}\min_{i=2,\ldots,d}\{|z_i-z_1|\}\right)^{d-1}=:C_1^{-1}.$$

We remark that this lower bound does not depend on n.

Thus, we get from (11) an upper bound for the "small" term $|z-\varphi_1|$, namely

$$C_1^{-1}|z-\varphi_1|<\alpha^{-n(1+\epsilon)}$$

and therefore

$$|z-\varphi_1|<C_1\alpha^{-n(1+\epsilon)}. \tag{13}$$

We are going to approximate z by a finite sum extracted from

$$\begin{aligned}\varphi_1 &= z_0+\sum_{|i|>0}a_{i,1}\gamma_1^{i_1 n}\cdots\gamma_r^{i_r n}=\\ &= z_0+\sum_{0<|i|\leq H}a_{i,1}\gamma_1^{i_1 n}\cdots\gamma_r^{i_r n}+\mathcal{O}\left(\max\{\gamma_1,\ldots,\gamma_r\}^{n(H+1)}\right).\end{aligned}$$

We define

$$V_n:=z_0+\sum_{0<|i|\leq H}a_{i,1}\gamma_1^{i_1 n}\cdots\gamma_r^{i_r n},$$

where $H\geq 1$ is an integer to be chosen later. We may write

$$V_n=z_0+\sum_{i=1}^{h}e_i\beta_i^n,$$

where $e_i\in\overline{\mathbb{Q}}$ and with $1>\beta_1>\ldots>\beta_h>0$ and with $h\leq H^r$. We enlarge K at once and assume that it contains all the coefficients e_i. Observe that we have

$$|\varphi_1-V_n|\leq C_2\max\{\gamma_1,\ldots,\gamma_r\}^{n(H+1)}=C_2C_3^{n(H+1)}, \tag{14}$$

with $C_3:=\max\{\gamma_1,\ldots,\gamma_r\}<1$.

We need an estimate for $\mathcal{H}(z)$ for later purposes. We have

$$\mathcal{H}(z) = \mathcal{H}\left(\frac{y}{\alpha^n}\right) \le \max\{\mathcal{H}(y), \mathcal{H}(\alpha^n)\} \le c\alpha^n, \tag{15}$$

where we have used that $\mathcal{H}(y) = |y| \le c\alpha^n$ (since $y \in \mathbb{Z}$) and that $\mathcal{H}(\alpha^n) \le \alpha^n$ too (since α to some power is an integer).

We choose H so that

$$C_3^{H+1}\alpha^2 < 1. \tag{16}$$

This is possible since $C_3 < 1$ and from now on H is fixed and therefore also h, e_i, β_i for $i = 1, \dots, h$ are fixed. Moreover, we choose a finite set S of absolute values of K so that it contains all infinite absolute values and we require that all $\alpha_i^{(j)}, i = 1, \dots, t^{(j)}, j = 1, \dots, d$ are S-units. In particular, with this choice all the β_i are S-units and the z are S-integers, the last fact following from the relation $y = \alpha^n z$ and the fact that the y are integers.

Last we need an upper bound for $\hat{\mathcal{H}}((-z, 1, \beta_1^n, \dots, \beta_h^n))$. First it is easy to see that we have

$$\mathcal{H}(\gamma_i) \le \alpha^d,$$

since we have

$$\gamma_i = \frac{\alpha_k^{(j)}}{\alpha^{d-j}}$$

for certain j and k. Therefore, we get

$$\mathcal{H}(\beta_i) \le \alpha^{dH} \text{ for all } i = 1, \dots, h.$$

Finally, we conclude

$$\hat{\mathcal{H}}((-z, 1, \beta_1^n, \dots, \beta_h^n)) = \mathcal{H}(z)\mathcal{H}(\beta_1)^n \cdots \mathcal{H}(\beta_h)^n \le c\alpha^{n(1+dHh)},$$

where we have used (15).

Now we are ready to apply Theorem 2 with $c_0 = -1, c_1 = z_0, c_i = e_{i-1}$ for $i = 2, \dots, h+1$. We put $w_0 = z, w_1 = 1$ and $w_i = \beta_{i-1}^n$ for $i = 2, \dots, h+1$. Since w_i are S-units for $i \ge 1$, assumption (i) of Theorem 2 is verified for any choice of $\epsilon' > 0$.

We proceed to verify assumption (ii) for all large n, provided we choose a small enough ϵ', which we will fix later, and put $\delta = (h+4)\epsilon'$. We have by (13) and (14)

$$\begin{aligned} &|c_0 w_0 + c_1 w_1 + \ldots + c_{h+1} w_{h+1}| = \\ &\quad = |z - V_n| \le |z - \varphi_1| + |\varphi_1 - V_n| \le C_1\alpha^{-n(1+\epsilon)} + C_2 C_3^{n(H+1)} \le \\ &\quad \le C_1\alpha^{-n(1+\epsilon)} + C_2\alpha^{-2n} \le C_4\alpha^{-n(1+\epsilon)}, \end{aligned}$$

where $C_3^{H+1} < \alpha^{-2}$ holds by (16).

To compare this estimate with the right side of (ii) in Theorem 2, we first observe that $\mathcal{H}_S(w_0)=1$, since $w_0=z$ is an S-integer. Therefore we get

$$\left(\mathcal{H}(w_0)\mathcal{H}_S(w_0)^{h+1}\right)^{-1}\hat{\mathcal{H}}(\mathbf{w})^{-\delta} > c^{-(1+\delta)}\alpha^{-n(1+\delta(1+dHh))}$$

where we have used

$$\begin{aligned}\mathcal{H}(w_0) &= \mathcal{H}(z) < c\alpha^n,\\ \hat{\mathcal{H}}(\mathbf{w}) &= \hat{\mathcal{H}}((-z,1,\beta_1^n,\dots,\beta_h^n)) \le c\alpha^{n(1+dHh)}.\end{aligned}$$

The verification of (ii) will follow from

$$C_4\alpha^{-n(1+\epsilon)} < c^{-(1+\delta)}\alpha^{-n(1+\delta(1+dHh))},$$

which is clearly true by choosing

$$\delta=(h+4)\epsilon' < \frac{\epsilon}{(1+dHh)}$$

and n large enough.

Now from Theorem 2 it follows that either there are finitely many possibilities for $\mathbf{w}=(-z,1,\beta_1^n,\dots,\beta_h^n)$ and therefore finitely many pairs (n,z) of solutions of (11), or we have a vanishing subsum of

$$c_0w_0+c_1w_1+\dots+c_{h+1}w_{h+1}=-z+z_0+\sum_{i=1}^h e_i\beta_i^n$$

involving $c_0w_0=-z$, i.e. we have

$$z=v_0+\sum_{i=1}^h v_i\beta_i^n,\quad v_i\in\overline{\mathbb{Q}}, i=0,\dots,h, \tag{17}$$

with $v_0\in\{0,z_0\}$ and $v_i\in\{0,e_i\}, i=1,\dots,h$. Substituting this into $|g(x_1,\dots,x_r,z)|$ we get

$$\left|g\left(\gamma_1^n,\dots,\gamma_r^n,v_0+\sum_{i=1}^h v_i\beta_i^n\right)\right| < \alpha^{-n(1+\epsilon)}$$

and we consequently see that this can either hold for all n in a suitable arithmetic progression $\mathcal{P}$ or holds only for finitely many n. The first case leads to the power sum

$$H_n=v_0\alpha^n+\sum_{i=1}^h v_i\left(\beta_i\alpha\right)^n,$$

having the property that (n,H_n) is a solution of (8) for all n in the above arithmetic progression. The second case leads again to finitely many solutions.

Finally, we get that all solutions of

$$\left|y^d+G_n^{(1)}y^{d-1}+\dots+G_n^{(d)}\right| < \alpha^{n(d-1-\epsilon)}$$

can be parametrized by finitely many power sums. Moreover, if we have infinitely many solutions then n lies in the union of a finite set and a finite number of arithmetic progressions. This proves the theorem. □

6. Proof of the Corollaries

The proof of Corollary 1 runs along the same lines as the proof of Corollary 3.3 in [18]. □

Corollary 2 is an immediate consequence of Theorem 1. □

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