On quantitative aspects of the unit sum number problem

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Abstract. We investigate the function $u_{K,S}(n;q)$, which counts the number of representations of algebraic integers α with $|N_{K/\mathbb{Q}}(\alpha)| \leq q$ for some real positive q, so that they can be written as sum of exactly n S-units of the number field K.

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1. Introduction

The study of the additive structure of units in rings originates from investigations due to Zelinsky [15] back in the fifties of the previous century. Zelinsky showed that every endomorphism of a vector space V over a division ring D can be written as the sum of two automorphisms, unless D is the field with two elements and V is of dimension 1. Similar results were obtained for other endomorphism rings (for an overview see [14]).

The additive unit structure of maximal orders of number fields has been studied, too. In particular, Jacobson [10] observed that every element in the maximal orders of the number fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ can be written as the sum of distinct units. Later all quadratic, cubic and quartic fields with this property were determined (see [12, 2, 3]). By a combination of results on S-unit equations and combinatorial results Jarden and Narkiewicz [11] showed that given a finitely generated domain B of characteristic zero then for every positive integer k there exists an $\alpha \in B$ such that α can not be written as the sum of k units. In view of this observation two problems arise:

The qualitative problem: Which ring of integers are generated by their units?

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The quantitative problem: How many non-associated algebraic integers with bounded norm, height, etc. can be written as the sum of k units?

The qualitative problem was solved for quadratic and complex cubic fields (see [2, 1, 13]). Also some special classes of quartic fields were investigated (see [9, 16]). Recently the quantitative problem was investigated for quadratic fields and some purely quartic fields (see [9, 8]).

This paper is devoted to the quantitative problem. Let K be a number field, $N_{K/\mathbb{Q}}$ the field norm and S a finite set of places of K including the archimedian ones. We denote by $\mathcal{O}_{K,S} = \{\alpha \in K : |\alpha|_\nu \leq 1 \text{ for all } \nu \notin S\}$ the ring of S-integers and its unit group by $U_{K,S}$; the group of S-units. As usual two S-integers α and β of K are said to be associated if there exists an S-unit ϵ such that $\alpha = \beta\epsilon$ and we write $\alpha \sim \beta$. In order to give precise statements we introduce the following counting function:

Definition 1. *The counting function $u_{K,S}(n;q) = u(n;q)$ is defined as number of equivalence classes $[\alpha]_\sim$ such that*

$$N(\alpha) := \prod_{\nu \in S} |\alpha|_\nu \leq q, \qquad \alpha = \sum_{i=1}^{n} \epsilon_i, \qquad \epsilon_i \in U_{K,S}$$

and no subsum of $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n$ vanishes.

Note that the function $u(n;q)$ is well defined, since $N(\alpha) = N(\beta)$ if $\alpha \sim \beta$. Before we state our main theorem we fix the following notation for the rest of the paper. Let $|S| = s+1$. Then the group $U_{K,S}$ of S-units of K is a free abelian group with s generators (see [4]) and we fix a fundamental system $\epsilon_1, \ldots, \epsilon_s$ of S-units generating it. By ω_K and $\mathrm{Reg}_{K,S}$ we will denote the number of roots of unity and the S-regulator of K, respectively.

Theorem 1. *Let $\epsilon > 0$. Under the above assumptions we have*

$$u(n;q) = \frac{c_{n-1,s}}{n!} \left(\frac{\omega_K (\log q)^s}{\mathrm{Reg}_{K,S}} \right)^{n-1} + o((\log q)^{(n-1)s-1+\epsilon})$$

as $q \to \infty$, where the constant $c_{n,s}$ is the volume of

$$\{(x_{11}, \ldots, x_{ns}) \in \mathbb{R}^{ns} \ : \ h(x_{11}, \ldots, x_{ns}) < 1\}$$

with

$$h(x_{11}, \ldots, x_{ns}) = \sum_{i=1}^{s} \max\{0, x_{1i}, \ldots, x_{ni}\} + \max\left\{0, -\sum_{i=1}^{s} x_{1i}, \ldots, -\sum_{i=1}^{s} x_{ni}\right\}.$$

Our theorem is closely related to results due to Everest [5, 6] who counted the number of solutions to certain norm inequalities. However, the full result of Everest makes heavy use of many non-elementary results such as theorems on the uniform distribution of sequences, Schmidt's subspace theorem, Baker's theory on linear forms in logarithms, etc. Using the full result would lead us to an error term without ϵ and a big-O instead of a small-o, where we use O and o with their usual meaning.

In order to give the reader more insights we sketch Everest's proof and try to avoid most of the non elementary steps. Unfortunately, we cannot avoid an application of Schmidt's subspace theorem (see Lemma 3, second statement). However, the interested reader is advised to study Everest's papers for a full account to the techniques described below.

In the next section we introduce some notations that will help us (in Section 3) to state Everest's results and to sketch his proofs. Applying these results we will prove our main theorem in Section 4. Since the definition of the constant $c_{n,s}$ is not very illuminating we discuss a formula for $c_{n,s}$ in the last section. This also enables us to compare the result with previous results of this form (particularly with the results in [8]).

2. Some Notations

We want to study S-unit representations with small norm and so we introduce the following functions. We will use the definition

$${}^*U_{K,S}^n = \{\underline{x} = (1, x_2, \ldots, x_n) \in U_{K,S}^n : \text{ no subsum of } 1 + x_2 + \cdots + x_n \text{ vanishes}\}.$$

Let $\underline{x} = (1, x_2, \ldots, x_n) \in U_{K,S}^n$. Then consider

$$N_{K,S}(\underline{x}) = N(\underline{x}) = \prod_{\nu \in S} |1 + x_2 + \cdots + x_n|_\nu$$

and its related counting function

$$N_{K,S}(q) = N(q) = \sharp \left\{ \underline{x} \in {}^*U_{K,S}^n \; : \; N(\underline{x}) < q \right\}.$$

We are also interested in the following variant: Let $\underline{c} = (c_1, \ldots, c_n) \in (K^*)^n$ be fixed, then we define

$$N_{\underline{c}}(\underline{x}) = \prod_{\nu \in S} |c_1 + c_2 x_2 + \cdots + c_n x_n|_\nu$$

and the corresponding counting function $N_{\underline{c}}(q)$ by the number of $\underline{x} \in U_{K,S}^n$ such that $N_{\underline{c}}(\underline{x}) < q$ and no subsum of $c_1 + c_2 x_2 + \cdots + c_n x_n$ vanishes.

As we will see in the next section these functions are closely related to the height functions

$$H_\nu(\underline{x}) = \max\{|x_i|_\nu : i = 1, \ldots, n\}$$

and

$$H(\underline{x}) = \prod_{\nu \in S} \max\{|x_i|_\nu : i = 1, \ldots, n\} = \prod_{\nu \in S} H_\nu(\underline{x})$$

and their counting function

$$H(q) = \sharp \left\{ \underline{x} \in U_{K,S}^n \; : \; H(\underline{x}) < q \right\}.$$

As will be shown later the following subset $U_0 \subset U_{K,S}^n$ will yield the main contribution to $H(q)$: We write H_ν^* to be the second largest member of the set $\{|x_i|_\nu : i = 0, \ldots, n\}$. Given real numbers $A_\nu > 0, B_\nu > 1$ we define

$$U_0 = \left\{ \underline{x} \in {}^*U_{K,S}^n \ : \ \forall \nu \in S, \frac{H_\nu^*(\underline{x})}{H_\nu(\underline{x})} < A_\nu e^{-B_\nu \log\log H(\underline{x})} \right\}.$$

Next we define the counting function

$$H_0(q) = \sharp \{ \underline{x} \in U_0 \ : \ H(\underline{x}) < q \}.$$

Assume for the rest of the paper that A_ν and B_ν are fixed numbers. The particular choice of A_ν and B_ν does not effect the asymptotics but rather the constants in the error terms, which are not the subject of interest in our considerations.

3. Sums of S-units

In this section we adapt the results and techniques due to Everest [5, 6] to our case. We start with

Lemma 1. *For $q \to \infty$ we have*

$$H(q) = c_{n-1,s} \left(\frac{\omega_K (\log q)^s}{\mathrm{Reg}_{K,S}} \right)^{n-1} + O((\log q)^{(n-1)s-1}).$$

Proof. Let $S = \{\nu_1, \ldots, \nu_{s+1}\}$ and $x_1 = 1, x_2, \ldots, x_n$ be S-units. Write $x_i = \zeta_i \epsilon_1^{k_{i1}} \cdots \epsilon_s^{k_{is}}$ for $i = 2, \ldots, n$. Taking logarithms of the absolute values $|\cdot|_\nu$ with $\nu \in S$ and recalling the definition of $H(\underline{x})$ we see that the function $H(q)$ essentially counts the number of points $\underline{k} = (k_{21}, \ldots, k_{ns}) \in \mathbb{Z}^{(n-1)s}$ that fulfill the inequalities

$$\sum_{j=1}^{s+1} \max\{L_i^{(j)}(\underline{k}) : i = 1, \ldots, n\} < \log q \tag{1}$$

for $i = 1, \ldots, n$, where $L_1^{(j)}(\underline{k}) = 0$ and

$$L_i^{(j)}(\underline{k}) = k_{i1} \log |\epsilon_1|_{\nu_j} + \cdots + k_{is} \log |\epsilon_s|_{\nu_j} = \log |x_i|_{\nu_j}$$

for $i = 2, \ldots, n$, and for $j = 1, \ldots, s+1$. A linear transformation of determinant $\mathrm{Reg}_{K,S}^{-1}$ turns the $L_i^{(j)}(\underline{k})$ into

$$\tilde{L}_i^{(j)}(\underline{k}) = \begin{cases} k_{ij} & \text{for } j \le s \\ -\sum_{l=1}^{s} k_{il} & \text{for } j = s+1. \end{cases}$$

Note that the number of lattice points in a polyhedron of dimension n is its volume plus an error term of the size about the $n-1$ dimensional volume of its faces. Bearing in mind the linear transformation and the fact that the number of lattice points are points counted by $H(q)$ up to multiplication by roots of unity we obtain the lemma. $\square$

For the function $H_0(q)$ we obtain (see [6, Proposition 1]):

Lemma 2. *For all $\epsilon > 0$ and $q \to \infty$ we have*

$$H_0(q) = H(q) + o\left((\log q)^{(n-1)s-1+\epsilon}\right).$$

Note that the proof of this lemma does not need any deep results and can be achieved by considering Dirichlet series (see also [5, Lemma 3]).

Next, we want to establish a relation between $H(\underline{x})$ and $N(\underline{x})$ (see [5, Lemma 6]).

Lemma 3. 1. *For all $\underline{x} \in U_0$ we have $\log N(\underline{x}) = \log H(\underline{x}) + O\left(1/\log(H(\underline{x}))\right)$.*
2. *For all $\underline{x} \in {}^*U_{K,S}$ there exists a constant $\theta > 0$ such that $\log N(\underline{x}) > \theta \log H(\underline{x})$.*

Proof. We have

$$\begin{aligned}\log N(\underline{x}) &= \sum_{\nu \in S} \log |1 + x_2 + \cdots + x_n|_\nu \\ &= \sum_{\nu \in S} \log H_\nu(\underline{x}) + \sum_{\nu \in S} \log \left(1 + O\left(\frac{H_\nu^*(\underline{x})}{H_\nu(\underline{x})}\right)\right).\end{aligned}$$

Because of the definition of U_0 the first statement is proved. The second statement is a consequence of Schmidt's subspace theorem (see [7, Theorem 2] and in the case of K a quadratic field see also [8, Lemma 3]) □

Now we establish the main result of this section (see also [6, Theorem 1]):

Proposition 1. *For all $\epsilon > 0$ and $q \to \infty$ we have*

$$N(q) = c_{n-1,s}\left(\frac{\omega_K (\log q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o((\log q)^{(n-1)s-1+\epsilon}).$$

Proof. We break up the counting function into two parts:

$$\begin{aligned}N(q) &= \sharp\{\underline{x} \in {}^*U_{K,S} \, : \, N(\underline{x}) < q, x \in U_0\} + \sharp\{\underline{x} \in {}^*U_{K,S} \, : \, N(\underline{x}) < q, x \notin U_0\} \\ &= N_1(q) + N_2(q)\end{aligned}$$

For the second part we obtain by Lemma 3

$$N_2(q) = \sharp\{\underline{x} \in {}^*U_{K,S} \, : \, H(\underline{x})^\theta < q, x \notin U_0\} = o\left((\log q)^{(n-1)s-1+\epsilon}\right),$$

where the last equality is obtained by a combination of the Lemmas 1 and 2. Since those $\underline{x}$ that satisfy $H(\underline{x}) < (\log q)^{1/2}$ are only $O((\log q)^{(n-1)s/2})$ in number, we may assume $H(\underline{x}) > (\log q)^{1/2}$. So we obtain

$$\begin{aligned} N_1(q) &= \sharp\{\underline{x} \in U_0 : \log(H(\underline{x})) < \log q + O(1/\sqrt{\log q})\} \\ &= H_0\left(q^{1+O(1/\sqrt{\log q})}\right) \\ &= H\left(q^{1+O(1/\sqrt{\log q})}\right) + o\left(((1+O(1/\sqrt{\log q}))\log q)^{s(n-1)-1+\epsilon}\right) \\ &= c_{n-1,s}\left(\frac{\omega_K(\log q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o((\log q)^{(n-1)s-1+\epsilon}) + O(\log q^{(n-1)s-3/2}) \\ &= c_{n-1,s}\left(\frac{\omega_K(\log q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + o((\log q)^{(n-1)s-1+\epsilon}). \end{aligned}$$

Thus the statement follows. □

Corollary 1. *Assume for all $\nu \in S$ that $\log|\epsilon_1|_\nu, \dots, \log|\epsilon_s|_\nu$ generate a $\mathbb{Q}$-space of dimension at least 2. Then*

$$N(q) = c_{n-1,s}\left(\frac{\omega_K(\log q)^s}{\mathrm{Reg}_{K,S}}\right)^{n-1} + O((\log q)^{(n-1)s-1}).$$

Proof. Combine Everest's result [6, Theorem 1] with Proposition 1. □

At the end of the section we want to state Everest's result [6, Theorem 1] for $N_{\underline{c}}(q)$ which stated in this form may be obtained analogously to the proof of Proposition 1, i.e. the only non elementary part is the proof of the second statement of Lemma 3.

Proposition 2. *For all $\underline{c} \in (K^*)^n$ we have*

$$N_{\underline{c}}(q) = O((\log q)^{(n-1)s}).$$

Now we have collected all results that are needed to prove the main theorem (see next section).

4. The number of S-unit representations

This section is devoted to the proof of our main theorem (Theorem 1). Assume $\alpha = x_1 + \ldots + x_n$ can be written as a sum of S-units, where

$$x_i = \zeta_i \epsilon_1^{k_{i1}} \cdots \epsilon_s^{k_{is}}$$

for $i = 1, \dots, n$. Then we can write $\alpha = 1 + x_2 + \cdots + x_n$ with $k_{11} = \ldots = k_{1s} = 0, k_{i1} \le \cdots \le k_{is}$ for $i = 2, \dots, n$ and $0 = k_{1s} \le k_{2s} \le \cdots \le k_{ns}$, since we are interested only in equivalence classes of associated integers. Let us denote by $u_0(n;q)$ the number of representations, where all units are pairwise distinct. Then we have $u_0(n;q) = N(q)/n! - r$, where r is the number of integers α that have two distinct representations as sums of n distinct S-units. As shown in [8, Lemma 2] r depends only on n and S but not on q. The factor $1/n!$ comes from the fact that the order of the x_i does not matter in the computation of $u_0(n;q)$.

On the other hand we have

$$u(n;q) = u_0(n;q) + \sum_{\underline{c}} \frac{N_{\underline{c}}(q)}{l(\underline{c})!} + O(1), \tag{2}$$

where the $O(1)$ comes from the α's with two distinct representations and the sum runs over all $\underline{c} = (c_1, \dots, c_m) \in \mathbb{Z}^m$ with $c_i \geq 1$ and $c_1 + \cdots + c_m = n$ and $(c_1, \dots, c_m) \neq (1, \dots, 1)$, i.e. $m < n$ for all $\underline{c}$'s appearing in the summation. Moreover we write $l(\underline{c}) = m$. Applying the Propositions 1 and 2 to formula (2) proves our main Theorem 1.

5. Some results for $c_{n,s}$

In this section we want to discuss the quantity $c_{n,s}$. For small values of s and n it is possible to compute the quantity $c_{n,s}$ directly. We did this for all pairs (n,s) with $n + s \leq 6$ (see Table 1).

TABLE 1. Values for $c_{n,s}$

$s \diagdown n$	1	2	3	4	5
1	2	3	4	5	6
2	3	15/4	7/2	45/16	
3	10/3	7/3	55/54		
4	35/12	275/32			
5	21/10				

It is also possible to calculate all values of $c_{n,s}$ for either $n = 1$ or $s = 1$. In general, we were not able (yet) to succeed in doing the necessary calculations. We can prove the following:

Lemma 4. *We have*

$$c_{n,1} = n + 1, \qquad c_{1,s} = \frac{1}{s!}\binom{2s}{s}$$

and

$$\frac{2^{ns}}{(ns)!} < c_{n,s} < 2^{ns}.$$

Proof. First, we consider the case $s = 1$, i.e. we have to compute the volume of

$$\mathcal{B}_{n,1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \ : \ \max\{0, x_1, \dots, x_n\} + \max\{0, -x_1, \dots, -x_n\} < 1\}.$$

Assume that the following holds:

$$x_1 > x_2 > \cdots > x_r > 0 > x_{r+1} > \cdots > x_n$$

with $0 < r < n$. Then we have $x_1 - x_n < 1$. Note that the case $r = 0$, respectively $r = n$, yields the volume of a simplex with rectangular edges of length 1 and hence volume $1/n!$. In the other cases the volume is

$$\begin{aligned}
&\int_{x_1=0}^{1}\int_{x_n=x_1-1}^{0}\int_{x_2=0}^{x_1}\cdots\int_{x_r=0}^{x_{r-1}}\int_{x_{n-1}=x_n}^{0}\cdots\int_{x_{r+1}=x_{r+2}}^{0} 1\,dx_{r+1}\cdots dx_{n-1}dx_r\cdots dx_2dx_ndx_1\\
&\qquad=\int_{x_1=0}^{1}\int_{x_n=x_1-1}^{0}\int_{x_2=0}^{x_1}\cdots\int_{x_r=0}^{x_{r-1}}\frac{(-x_n)^{n-r-1}}{(n-r-1)!}\,dx_r\cdots dx_2dx_ndx_1\\
&\qquad=\int_{x_1=0}^{1}\int_{x_n=x_1-1}^{0}\frac{(-x_n)^{n-r-1}x_1^{r-1}}{(n-r-1)!(r-1)!}\,dx_ndx_1\\
&\qquad=\int_{x_1=0}^{1}\frac{(1-x_1)^{n-r}x_1^{r-1}}{(n-r)!(r-1)!}dx_1=\frac{1}{n!}.
\end{aligned}$$

So any case (also the cases $r = 0$ and $r = n$) yields the volume $1/n!$. Since there are $(n + 1)!$ cases we obtain $c_{n,1} = (n + 1)!/n! = n + 1$.

Now we consider the case $n = 1$. In this case we have to compute the volume of

$$\mathcal{B}_{1,s} := \{(x_1, \ldots, x_s) \in \mathbb{R}^s \,:\, h(x_1, \ldots, x_s) < 1\},$$

where $h(x_1, \ldots, x_s) := \max\{0, x_1\} + \cdots + \max\{0, x_s\} + \max\{0, -x_1 - \cdots - x_s\}$. Let us assume $x_1, \ldots, x_r > 0$ and $x_{r+1}, \ldots, x_s \le 0$. Then $h(x_1, \ldots, x_s) = x_1 + \cdots + x_r$ if $-x_1 - \ldots - x_s \le 0$ and $h(x_1, \ldots, x_s) = -x_{r+1} - \cdots - x_s$ otherwise. In the second case we may make the coordinate change $y_i = -x_i$ with $1 \le i \le s$ and obtain the first case with $s - r$ instead of r. Therefore we consider only the case $x_1, \ldots, x_r > 0$, $x_{r+1}, \ldots, x_s \le 0$ and $x_1 + \ldots + x_s > 0$. Then we have to compute the following integral:

$$\begin{aligned}
&\int_{x_1=0}^{1}\int_{x_2=0}^{1-x_1}\cdots\int_{x_r=0}^{1-\sum_{i=1}^{r-1}x_i}\ \int_{x_{r+1}=-\sum_{i=1}^{r}x_i}^{0}\cdots\int_{x_s=-\sum_{i=1}^{s-1}x_i}^{0} 1\,dx_s\cdots dx_1\\
&\qquad=\int_{x_1=0}^{1}\cdots\int_{x_r=0}^{1-\sum_{i=1}^{r-1}x_i}\frac{\left(\sum_{i=1}^{r}x_i\right)^{s-r}}{(s-r)!}dx_r\cdots dx_1\\
&\qquad=\int_{x_1=0}^{1}\cdots\int_{x_{r-1}=0}^{1-\sum_{i=1}^{r-2}x_i}\frac{1}{(s-r+1)!}-\frac{\left(\sum_{i=1}^{r-1}x_i\right)^{s-r+1}}{(s-r+1)!}dx_{r-1}\cdots dx_1
\end{aligned}$$

$$= \frac{1}{(s-r+1)!(r-1)!} - \int\limits_{x_1=0}^{1} \cdots \int\limits_{x_{r-1}=0}^{1-\sum_{i=1}^{r-2} x_i} \frac{(\sum_{i=1}^{r-1} x_i)^{s-r+1}}{(s-r+1)!} dx_{r-1} \cdots dx_1$$

$$= \sum_{j=1}^{r} \frac{(-1)^{j+1}}{(s-r+j)!(r-j)!} = \frac{1}{(s-r)!(r-1)!s}.$$

All together we obtain

$$c_{n,s} = \sum_{r=0}^{s} \frac{2\binom{s}{r}}{(s-r)!(r-1)!s} = \frac{4^s\Gamma(s+1/2)}{\sqrt{\pi}\Gamma(s+1)^2} = \frac{2\Gamma(2s)}{s!^2\Gamma(s)} = \frac{(2s)!}{s!^3} = \frac{1}{s!}\binom{2s}{s},$$

where we have used well-known summation formulas for hypergeometric sums and the duplication formula for the Gamma function.

We are left to prove the last statement of the lemma. Since we have $|x_{ij}| < 1$ for $1 \le i \le n$ and $1 \le j \le s$ the upper bound is established. For the lower bound we use the fact that all x_{ij} satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{s} |x_{ij}| < 1$$

lie in the body that defines $c_{n,s}$. $\square$

Remark 1. With a little more effort it would be possible to show sharper bounds for $c_{n,s}$. We abandon to do so because the constant does not effect the asymptotics of $u(n;q)$ which is our main concern in this paper.

Let us consider the result for $s = 1$, i.e. let us consider the unit group of a real quadratic field, complex cubic field or totally complex quartic field K. Then Theorem 1 yields

$$u(n;q) = \frac{1}{(n-1)!}\left(\frac{\omega_K(\log q)}{\log|\eta|}\right)^{n-1} + o((\log q)^{(n-2+\epsilon}),$$

where η is a fundamental unit of K with $|\eta| > 1$. Moreover, note that the number of roots of unity $\omega_K = 2$ if K is quadratic or a cubic field or a quartic field containing neither i nor $\frac{1+i\sqrt{3}}{2}$. In particular, for K a real quadratic field we get

$$u(n;q) = \frac{1}{(n-1)!}\left(\frac{2(\log q)}{\log \eta}\right)^{n-1} + o((\log q)^{(n-2+\epsilon}),$$

which is an improvement to the result obtained in [8].

Moreover, we have an asymptotic formula with explicit main term also for $n = 2$ and arbitrary s. In the general case we now have the leading term but without an explicit constant.

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