1. Dimenson data

Definition 1. *We call the **dimension data** for $H \subset G$ the data associating*

$$\dim V^H$$

to every finite dimensional complex representaion V of G.

$H_1, H_2 \subset G$ are said to possess the same dimension data if

$$\dim V^{H_1} = \dim V^{H_2}$$

for any finite dimensional representation V of G.

For a faithful representation (ρ, V) of a Lie group G, we have the data of the invariant dimensions of various tensors of V, which is identical to the dimension data of the inclusion $G \subset GL(V)$.

The dimension data problem asks, *"what we can say for two groups H_1, H_2 if they have inclusions to a group G with the same dimension data ?"*

The native hope is to show $H_1 \sim H_2$ (or $(G_1, \rho_1, V_1) \cong (G_2, \rho_2, V_2)$) provided that $H_1, H_2 \subset G$ (or (G_1, ρ_1, V_1), (G_2, ρ_2, V_2)) have the same dimension data. But the conjugacy relation doesn't hold in some examples. So one should try to show a weaker relation or to consider the dimension data problem under some additional conditions.

2. Dade 's example

In [D], Dade answered a question of R. Brauer negatively by giving two 3-generator, 3-step nilpotent, exponent p, order p^7 non-isomorphic finite groups G, G^* with identical character table and satisfying all other additional conditions proposed by Brauer.

In Dade 's example, let $H_1 = G$, $H_2 = G^*$, take an i such that $\chi_i(1) = p^3$, let ρ_1, ρ_2 be representations with characters χ_i, χ_i^* respectivly. Then $(H_1, \rho_1), (H_2, \rho_2)$ have the same dimension data.

3. Larsen-Pink 's main results

In [LP], Professors M. Larsen and R. Pink worked on dimension data problem for faithful complex representations of complex semisimple linear groups.

Theorem 1([LP]) *For any faithful finite dimensional representation of a connected semisimple Lie group G, dimension data uniquely determines G up to isomorphism.*

Theorem 2([LP]) *Under the hypotheses of Theorem 3, if ρ is irreducible, dimension data uniquely determines ρ up to isomorphism.*

Theorem 3([LP]) *In the full generality of Theorem 3, ρ is not determined up to isomorphism by dimension data.*

Example 2. *In pages 392-393 of* [LP]*, starting from a pair of isomorphic but non-conjugate sub-root systems $\Phi_1, \Phi_2 \subset \Psi = r(BC_n)$, they produced a semisimple compact connected Lie group G and two non-isomorphic representations (ρ, ρ') with the same dimension data.*

It is remarked in [LP] *that the smallest rank of G constructed in this way is* 78 *and the smallest dimension is rather large.*

4. Motivation and connections

It is said in [LP] that their work on dimension data was motivated from a "Tannakian" type question: to what extent is a complex linear Lie group, G, and a finite dimensional representation, (ρ, V) of G, determined by the dimensions of the various invariant spaces W^G, where W is obtained from V by linear algebra(tensor, symmetry tensor, anti-symmetric tensor, etc).

I learned the dimension data problem from Professors Jinpeng An and Jiukang Yu. They are interested on *"whether the dimension data determine semisimplicity ?"* (that is, if $H_1, H_2 \subset G$ have the same dimension data and one of H_1, H_2 is semisimple, is another one of H_1, H_2 also semisimple ?)

The general *branching rule problem* asks, "how each irreducible finite dimensional representation of a group G decompose when it is viewed as a representation of a subgroup H by restriction ?"

In branching rule problem for $H \subset G$, when we look at the multiplicities of the trivial representation of H appearing in representations of G, we just get dimension data. So dimension data is part of information contained in branching rule.

The dimension data also arises in other aspects of mathematics.

- In [LP2], it arised in Larsen-Pink 's study of compatible systems of l-adic Galois representations.
- In [S], C. Sutton used the counter-examples in [LP] to produce examples of isospectral non-isomorphic simply connected Riemannian manifolds.
- In [L], R. Langlands proposed a connection between dimension data and the orders of the pole of L-functions at $s = 1$. For this, we recommend you to have a look at Langlands' paper [L] or [L2] for his idea.

5. Counter examples of An-Yu-Yu

In [AYY], we get a class of non-isomorphic representations with the same dimension data.

Theorem 3. *For each $n \geq 1$, let $G = SU(4n+2)$,*

$$H_1 = \{\mathrm{diag}\{A, \bar{A}\} \in SU(4n+2) | A \in U(2n+1)\}$$

and

$$H_2 = \{\mathrm{diag}\{A, B\} \in SU(4n+2) | A \in Sp(n), B \in SO(2n+2)\}.$$

Then $H_1, H_2 \subset G$ possess the same dimension data but $H_1 \ncong H_2$.

One can see that, in the above example, for any $n \geq 1$, $H_2 \cong Sp(n) \times SO(2n+2)$ is semisimple but $H_1 \cong U(2n+1)$ is not, so dimension data doesn't determine semisimplicity.

Use a way of construction of isospectral manifolds in [S], our examples lead to examples of pairs of isospectral simply connected but non-homeomorphic manifolds.

Corollary 4. *For each $n \geq 1$, the Riemannian manifolds $X_n = SU(4n+2)/U(2n+1)$ and $Y_n = SU(4n+2)/(Sp(n) \times SO(2n+2))$ with metrics induced from a bi-invariant metric on $SU(4n+2)$ are simply connected and isospectral but non-homeomorphic.*

Note that,

$$H_2(X_n, \mathbb{Z}) = \mathbb{Z}, H_2(Y_n, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z},$$

so X_n, Y_n have different homology.

6. Conventions

A *root system* is a finite set Φ consisting of non-zero vectors in an Euclidean linear space V, which is stable under the action of the reflection s_α for any $\alpha \in \Phi$. Recall that

$$s_\alpha(v) = v - \frac{2\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha, \forall v \in V,$$

and the group W_Φ generated by the reflections $s_\alpha : \alpha \in \Phi$ is called the Weyl group of Φ. Φ is called reduced if for any $\alpha \in \Phi$, $\frac{1}{2}\alpha \notin \Phi$. A subset Φ' of Φ is called a *sub-root system* if Φ' is itself a root system.

Each root system is the direct sum of simple root systems, and simple root systems can be classified into types $\{A_n, B_n, C_n, D_n : n \geq 1\}$(with some repetition in lower rank) and $\{E_6, E_7, E_8, F_4, G_2\}$.

Choose an ordering on V, the positive vectors in Φ consist in the positive system Φ^+, let

$$\delta_\Phi = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha, \quad A_\Phi = \frac{1}{|W_\Phi|}\sum_{w\in W_\Phi}[\delta_\Phi - w\delta_\Phi].$$

For any finite group Γ between W_Φ and $O(V)$, define

$$F_{\Phi,\Gamma} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma(A_\Phi) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}[\delta_\Phi - \gamma\delta_\Phi].$$

7. The classification of counter-examples

We showed the dimension data problem can be reduced the comparison of characters associated to two sub-root systems in a root system. For the latter, we have the following answer.

Theorem 5. *For a simple root system Ψ_0, if there exists non conjugate sub-root systems $\Phi_1, \Phi_2 \subset \Psi_0$ such that $F_{\Phi_1,W_{\Psi_0}} = F_{\Phi_2,W_{\Psi_0}}$. Then $\Psi_0 = C_n, BC_n, F_4$.*

When $\Psi_0 = C_n, BC_n$, $F_{\Phi_1,W_{\Psi_0}} = F_{\Phi_2,W_{\Psi_0}}$ if and only if

$$\forall m \leq n, b_m(\Phi_1) - b_m(\Phi_2) = a_{2m}(\Phi_1) - a_{2m}(\Phi_2) = 0,$$

$$\text{and } \forall m \leq n, a_{2m+1}(\Phi_1) - a_{2m+1}(\Phi_2) = c_m(\Phi_2) - c_m(\Phi_1) = d_m(\Phi_2) - d_m(\Phi_1).$$

When $\Psi_0 = F_4$, $F_{\Phi_1,W_{\Psi_0}} = F_{\Phi_2,W_{\Psi_0}}$ if and only if $\Phi_1 \sim \Phi_2$,

$$\text{or } \{\Phi_1, \Phi_2\} \sim \{A_2^S, A_1^L + 2A_1^S\}, \{A_1^L + A_2^S, 2A_1^L + 2A_1^S\}.$$

Theorem 6. *Let $\Psi = \bigoplus_{1\leq i\leq m}\Psi_i$ be the direct sum of simple root systems $\{\Psi_i\}_1^m$ and $\Phi_1, \Phi_2 \subset \Psi$. Then $F_{\Phi_1,\mathrm{Aut}(\Psi)} = F_{\Phi_2,\mathrm{Aut}(\Psi)}$ if and only if there exists $\gamma \in \mathrm{Aut}(\Psi)$ such that*

$$F_{\Phi_i^{(1)},W_{\Psi_i}} = F_{\Phi_i^{(2)},W_{\Psi_i}}, \forall i, 1 \leq i \leq m,$$

where $\gamma\Phi_1 = \bigoplus_{1\leq i\leq m}\Phi_i^{(1)}$ and $\Phi_2 = \bigoplus_{1\leq i\leq m}\Phi_i^{(2)}$.

8. Method of the proof

Part 1: *Reduction.*

Let $d\mu_H$ be a Haar measure on H normalized so that $\int_H 1 d\mu_H = 1$, $p_G : G \longrightarrow G^\#$ be the map of projection to conjugacy class, $(p_G)_* i_*(d\mu_H)$ is called the Sato-Tate measure.

Choose maximal tori T, S of H, G respectively with $T \subset S$ and let

$$\Lambda = \mathrm{Hom}(T, U(1)), \Gamma^0 = N_G(T)/C_G(T), \Gamma = \mathrm{Aut}(T) = \mathrm{Aut}(\Lambda).$$

From the calculation of Sato-Tate measure , Larsen-Pink was able to show *the dimension data determines conjugacy class of* T *and the character* F_{Φ,Γ°*, and vice-versa.*(this statement is slightly different with that in [LP])

Consider all sub-root systems $\Phi \subset \Lambda$ with $F_{\Phi,\Gamma} = F$, Larsen-Pink was abel to show *the existence of a unique maximal sub-root system* $\Psi \subset \Lambda$ *containing all such* Φ. Moreover Ψ is Γ stable and $\Gamma = \mathrm{Aut}(\Psi, \Lambda)$.

Then we are led to the following the question, which can be reduced the case to Ψ is either reduced or isomorphic to a BC_n.

Question 7. *Fix a root system* Ψ*, a lattice* Λ *with* $\mathbb{Z}\Psi \subset \Lambda \subset \Lambda_\Psi$*,* $\Gamma = \mathrm{Aut}(\Psi, \Lambda)$*, classify pairs of non-conjugate sub-root systems* $\Phi_1, \Phi_2 \subset \Psi$ *with* $F_{\Psi_1,\Gamma} = F_{\Phi_2,\Gamma}$*.*

Part 2: *Semisimple case.*

When Ψ is reduced, Larsen-Pink showed *the characters* $F_{\Phi,\Gamma}$ *are actually linearly independent for any system of pairwise non-conjugate sub-root systems.*

When $\Psi = BC_n$, Larsen-Pink showed the following.

Proposition 8. *the character rings of* $\{BC_n : n \geq 1\}$ *form a direct system and the direct limit is isomorphic to* $\mathbb{Q}[x_1, x_2, ..., x_n, ...]$*.*

Let b_k, c_k, d_k be the polynomials from sub-root systems B_k, C_k, D_k respectively.

Proposition 9. (1) $c_n, d_{n+1} \in \mathbb{Q}[x_1, x_2, ..., x_{2n}] - \mathbb{Q}[x_1, x_2, ..., x_{2n-1}]$, $b_n \in \mathbb{Q}[x_1, x_2, ..., x_{2n-1}] - \mathbb{Q}[x_1, x_2, ..., x_{2n-2}]$*, they are of constant term 1 and have integer coefficients.*

(2) *Each of* $\{b_n, c_n, d_{n+1} | n \geq 1\}$ *is a prime in* $\mathbb{Q}[x_1, x_2, ...]$ *and any two of them are different.*

(3) *Each of the subset* $\{b_1, ..., b_n, c_1, ..., c_n\}$*,* $\{b_1, ..., b_n, d_2, ..., d_{n+1}\}$*,* $\{c_1, ..., c_n, d_2, ..., d_{n+1}\}$ *is algebraicly independent.*

Part 3: *Reductive case.*

By Theorem 3, the fundament Theorem 1 in [LP] fails in reductive case, so the best hope is to answer the following question affirmatively.

Question 10. *If two sub-root systems have equal* Γ*-traces of characters, whether the characters of sub-root systems become equal after replace one of them by a* Γ *conjugate sub-root system?*

The answers to this question is given in Theorem 5 and 6.

When $\Psi = A_{n-1}$, the linear independence still holds, so one just shows as that in [LP].

When Ψ is of type B, C, D or non-reduced, apart from b_k, c_k, d_k, we define a polynomial a_k from the sub-root system A_{k-1}. Besides those conclusions already showed in [LP], the main technical step is to show $a_{2n} = b_n\sigma(b_n)$ and $a_{2n+1} = c_n d_{n+1}$.

Actually the polynomials turn out to be the determinants of some matrices. This fact is crucial to show $a_{2n} = b_n\sigma(b_n)$ and $a_{2n+1} = c_n d_{n+1}$.

When Ψ is exceptional, we showed for any dominant integeral weight λ, the different characters with λ as a leading term are linearly independent.

These are enough to prove Theorems 5 and 6.

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Department of Mathematics, ETH Zürich, 8092 Zürich, Switzerland

E-mail address: jun.yu@math.ethz.ch