7.6 Weil Restriction

The main purpose of this section is to discuss a criterion for the existence of Weil restrictions and to study the behavior of Néron models with respect to Weil restrictions.

Let $h : S' \longrightarrow S$ be a morphism of schemes. Then, for any S'-scheme X', the contravariant functor

$$\mathfrak{R}_{S'/S}(X') : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto \mathrm{Hom}_{S'}(T \times_S S', X') ,$$

is defined on the category (Sch/S) of S-schemes. If it is representable, the corresponding S-scheme, again denoted by $\mathfrak{R}_{S'/S}(X')$, is called the *Weil restriction* of X' with respect to h. Thus, the latter is characterized by a functorial isomorphism

$$\mathrm{Hom}_S(T, \mathfrak{R}_{S'/S}(X')) \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', X')$$

of functors in T where T varies over all S-schemes. There are several elementary properties of the functor $\mathfrak{R}_{S'/S}(X')$ and, hence, of Weil restrictions, which follow immediately from the definition. We will derive some of them once we have mentioned the adjunction formula in Lemma 1 below.

Imposing an appropriate condition on h such as being finite and locally free (which we mean as a synonym for being finite, flat, and of finite presentation), the existence of the Weil restriction of the affine n-space $\mathbb{A}^n_{S'}$ is trivial (cf. the beginning of the proof of Theorem 4). Then, in order to treat more general schemes, it is necessary to study the behavior of Weil restrictions with respect to open or closed immersions. In order not to worry about the representability of the functor $\mathfrak{R}_{S'/S}(X')$ too much, we will work entirely within the context of functors from schemes to sets. In particular, we will make no difference between an S-scheme X and its associated functor $\mathrm{Hom}_S(\cdot, X)$; in the same way we will proceed with S'-schemes.

It is convenient to define the functor $\mathfrak{R}_{S'/S}(X')$ not only for S'-schemes X', but, more generally, for arbitrary contravariant functors from the category (Sch/S') of S'-schemes to the category of sets. So consider a functor

$$F' : (\mathrm{Sch}/S')^0 \longrightarrow (\mathrm{Sets}) .$$

Then its direct image with respect to $h : S' \longrightarrow S$ consists of the functor

$$h_* F' : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets}) , \qquad T \longmapsto F'(T \times_S S') .$$

Using 4.1/1, we see easily that the functor

$$(\mathrm{Sch}/S) \longrightarrow (\mathrm{Sch}/S') , \qquad T \longmapsto T \times_S S' ,$$

plays the role of an adjoint of h_*; namely, the so-called adjunction formula is valid.

Lemma 1. *For any S-scheme T and any functor* $F' : (\mathrm{Sch}/S')^0 \longrightarrow (\mathrm{Sets})$, *there is a canonical bijection*

$$\mathrm{Hom}_S(T, h_* F') \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', F')$$

which is functorial in T and in F'.

As an application of the above formula, we want to derive some elementary properties of Weil restrictions. Let X' be an S'-scheme. Then the identity on $\mathfrak{R}_{S'/S}(X')$ gives rise to a functorial morphism

$$\mathfrak{R}_{S'/S}(X') \times_S S' \longrightarrow X'$$

if $\mathfrak{R}_{S'/S}(X')$ exists as an S-scheme. Likewise, if X is an S-scheme, the identity on $X \times_S S'$ defines a functorial morphism

$$X \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S') .$$

On the other hand, each functorial morphism $F' \longrightarrow G'$ between contravariant functors from (Sch/S') to (Sets) induces a functorial morphism $h_* F' \longrightarrow h_* G'$. Furthermore, h_* commutes with fibred products, and it follows that $h_* F'$ is a group functor if the same is true for F'. In particular, the Weil restriction of a group scheme is, if it exists as a scheme, a group scheme again. Also it is easy to see that the notion of Weil restriction is compatible with base change; i.e., if $T \longrightarrow S$ is a morphism of base change, and if we write $T' := S' \times_S T$, then, for any S'-scheme X', there is a canonical isomorphism

$$\mathfrak{R}_{T'/T}(X' \times_S T') \simeq \mathfrak{R}_{S'/S}(X') \times_S T$$

of functors on (Sch/T).

In the following we need the terminology of relative representability of functors; cf. Grothendieck [1], Sect. 3. Let

$$F, G : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$$

be contravariant functors, and let $u : F \longrightarrow G$ be a functorial morphism. Then, for each functorial morphism $T \longrightarrow G$, where T is an arbitrary S-scheme, the fibred product $F_T = F \times_G T$ may be viewed as a functor from $(\mathrm{Sch}/T)^0$ to (Sets). One says that F is *relatively representable* over G via u if, for each $T \longrightarrow G$, the projection $F_T \longrightarrow T$ is a morphism in (Sch/S); i.e., if each F_T is representable by a T-scheme. Many notions on morphisms between schemes can easily be adapted to the context of relative representability. For example, u is called an open immersion, or a closed immersion, or a morphism of finite type, etc., if the corresponding property is true for each morphism of schemes $u_T : F_T \longrightarrow T$, obtained from $u : F \longrightarrow G$ by the "base change" $T \longrightarrow G$.

Proposition 2. *Let $u' : F' \longrightarrow G'$ be a morphism between functors from* $(\mathrm{Sch}/S')^0$ *to* (Sets).

(i) *Assume that u' is an open immersion and that $h : S' \longrightarrow S$ is proper. Then the associated morphism $h_*(u') : h_* F' \longrightarrow h_* G'$ is an open immersion.*

(ii) *Assume that u' is a closed immersion and that $h : S' \longrightarrow S$ is finite and locally free or, more generally, proper, flat, and of finite presentation. Then $h_*(u') : h_* F' \longrightarrow h_* G'$ is a closed immersion.*

Proof. Let us write $F = h_* F'$ and $G = h_* G'$, and let $T \longrightarrow G$ be a morphism, where T is an arbitrary S-scheme. Setting $T' := T \times_S S'$, we claim that $T \longrightarrow G$ factors canonically through $h_* T'$. Indeed, we have a canonical morphism $T \longrightarrow h_* T'$.

Furthermore, $T \longrightarrow G$ corresponds to a morphism $T' \longrightarrow G'$ and, hence, to a morphism $h_* T' \longrightarrow h_* G' = G$. That the composition with $T \longrightarrow h_* T'$ yields $T \longrightarrow G$ is easily verified with the help of 4.1/1. Consequently, we can view F_T as being obtained from $F_{h_* T'}$ by means of the base change $T \longrightarrow h_* T'$, a fact to be used below.

Furthermore, since h_* commutes with fibred products, there are isomorphisms

$$h_* F'_{T'} \simeq F \times_G h_* T' \simeq F_{h_* T'} ,$$

and we can look at the canonical commutative diagram

$$\begin{array}{ccc} F'_{T'} & \longrightarrow & T' \\ & & \downarrow \\ F_T & \longrightarrow & T \\ \downarrow & & \downarrow \\ F_{h_* T'} & \longrightarrow & h_* T' \end{array} .$$

In order to prove assertion (i), it has to be shown that the morphism in the middle row, which is obtained from the one in the lower row by the base change $T \longrightarrow h_* T'$, is an open immersion of schemes. We know already that the upper row is an open immersion of schemes; let U' be the image of $F'_{T'}$ in T', and set $V' := T' - U'$. Then V' is closed in T' and, since $T' \longrightarrow T$ is proper, its image V in T is closed again. Set $U := T - V$. Interpreting F_T as the fibred product of $F_{h_* T'}$ and T over $h_* T'$, we have

$$F_T = \operatorname{Hom}_{S'}(\cdot \times_S S', U') \times_{\operatorname{Hom}_{S'}(\cdot \times_S S', T')} \operatorname{Hom}_S(\cdot, T) .$$

Thus, if Z is an arbitrary S-scheme, $F_T(Z)$ consists of all S-morphisms $Z \longrightarrow T$ where $Z \times_S S' \longrightarrow T'$ factors through U'; i.e., of those S-morphisms $Z \longrightarrow T$ which factor through U. Hence F_T is represented by the open subscheme U of T and assertion (i) follows.

Next, let us verify assertion (ii) for the case where h is finite and locally free. Similarly as before, let V' be the closed subscheme of T' which is given by the closed immersion $F'_{T'} \longrightarrow T'$. Then we have to find a closed subscheme V of T such that, given any S-morphism $Z \longrightarrow T$, it factors through V if and only if $Z \times_S S' \longrightarrow T'$ factors through V'. The problem is local on S, T, and Z, so we may assume that all three schemes are affine, say with rings of global sections R, A, and C. Let $R \longrightarrow R'$ be the homomorphism between rings of global sections on S and S'. We may assume R' is a free R-module of rank n. Let $e_1, \ldots, e_n$ be a basis of R' over R; then these elements give rise to a basis of $A \otimes_R R'$ over R. Furthermore, let $\mathfrak{a}' \subset A \otimes_R R'$ be the ideal corresponding to V', and fix generators a'_i, $i \in I$, of $\mathfrak{a}'$. There are equations

$$a'_i = \sum_{j=1}^{n} c_{ij} e_j , \qquad i \in I ,$$

with coefficients $c_{ij} \in A$. These coefficients generate an ideal $\mathfrak{a} \subset A$, and we claim that the associated closed subscheme $V \subset T$ is as required. Namely, consider the homomorphism $\sigma : A \longrightarrow C$ which is associated to $Z \longrightarrow T$ as well as the homomorphism $\sigma' : A \otimes_R R' \longrightarrow C \otimes_R R'$ associated to $Z \times_S S' \longrightarrow T'$. Since

$$\ker \sigma' = (\ker \sigma) \otimes_R R' = \bigoplus_{i=1}^{n} (\ker \sigma) \cdot e_i ,$$

we see that $\mathfrak{a}' \subset \ker \sigma'$ if and only if $\mathfrak{a} \subset \ker \sigma$, i.e., that Z' is mapped into V' if and only if Z is mapped into V. So it follows that V represents the functor F_T.

If, more generally, h is proper, flat, and of finite presentation, one uses techniques from the construction of Hilbert schemes as in [FGA], n°221, Sect. 3, in order to show that there is a largest closed subscheme V of T such that an S-morphism $Z \longrightarrow T$ factors through V if and only if, after base change with $h : S' \longrightarrow S$, it factors through $V' \subset T'$. □

A functor $F : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is called a *sheaf with respect to the Zariski topology* (see 8.1) if, for each S-scheme T and for each covering $\{T_i\}$ of T, the sequence

$$\mathrm{Hom}_S(T, F) \longrightarrow \prod_i \mathrm{Hom}_S(T_i, F) \rightrightarrows \prod_{i,j} \mathrm{Hom}_S(T_i \cap T_j, F)$$

is exact. Of course, if F is a scheme, F is a sheaf in this sense.

Proposition 3. *If $F' : (\mathrm{Sch}/S)^0 \longrightarrow (\mathrm{Sets})$ is a sheaf with respect to the Zariski topology, then the same is true for $F := h_* F'$.*

Proof. Since, for any S-scheme T, we have

$$\mathrm{Hom}_S(T, F) = \mathrm{Hom}_{S'}(T \times_S S', F') ,$$

the assertion is obvious. □

We want to apply the above results to the case where F' consists of an S'-scheme X', and give a criterion of Grothendieck for the representability of $X := h_* X' = \mathfrak{R}_{S'/S}(X')$ by an S-scheme. Then, if X is representable, it defines the Weil restriction of X'.

Theorem 4. *Let $h : S' \longrightarrow S$ be a morphism of schemes which is finite and locally free, and let X' be an S'-scheme. Assume that, for each $s \in S$ and each finite set of points $P \subset X' \otimes_S k(s)$, there is an affine open subscheme U' of X' containing P. Then $h_* X' = \mathfrak{R}_{S'/S}(X')$ is representable by an S-scheme X and, thus, the Weil restriction of X' exists.*

Proof. We may assume that S and, hence, S' are affine, say with rings of global sections R and R' and that R' is a free R-module, say with generators $e_1, \ldots, e_n$. Let us first show that $h_* X'$ is representable if X' is affine. So assume X' is affine and view it as a closed subscheme of some scheme $\mathrm{Spec}\, R'[t]$, where t is a (finite or infinite) system of indeterminates. Applying Proposition 2, it is only necessary to consider the case where $X' = \mathrm{Spec}\, R'[t]$. Consider n copies of the system t and write $t_1, \ldots, t_n$ for these systems. Then, for any R-algebra A, there is a bijection

$$\mathrm{Hom}_{R'}(R'[t], A \otimes_R R') \longrightarrow \mathrm{Hom}_R(R[t_1, \ldots, t_n], A) ,$$

which is functorial in A. In order to define this map, consider an R'-homomorphism $\sigma' : R'[t] \longrightarrow A \otimes_R R'$. The latter is determined by the image $\sigma'(t)$ of t in $A \otimes_R R'$. Using the direct sum decomposition

$$A \otimes_R R' = \bigoplus_{i=1}^{n} (A \otimes_R R)e_i \,,$$

we can write

$$\sigma'(t) = \sum_{i=1}^{n} \sigma(t_i) \otimes e_i$$

with systems $\sigma(t_1), \ldots, \sigma(t_n)$ of elements in A, and we can think of σ as of a homomorphism $\sigma : R[t_1, \ldots, t_n] \longrightarrow A$. Then it is easily seen that $\sigma' \longmapsto \sigma$ defines the desired bijection. Consequently, in this case the functor $h_* X'$ is representable by the S-scheme Spec $R[t_1, \ldots, t_n]$, and it follows that the Weil restriction $\mathfrak{R}_{S'/S}(X')$ exists.

Next, let us consider the case where X' is not necessarily affine. Let $\{U_i'\}_{i \in I}$ be the system of all affine open subschemes of X'. Then, by what we have just seen, each $h_* U_i'$ is representable by an (affine) scheme U_i, and the open immersion $U_i' \hookrightarrow X'$ gives rise to a morphism $U_i \longrightarrow h_* X'$ which is an open immersion by Proposition 2. Viewing the U_i' as open subschemes of X', we have canonical gluing data for them, and these data give rise to gluing data for the U_i. So, gluing the U_i, we obtain an S-scheme Y. Since X' is a sheaf with respect to the Zariski topology, the same is true for $h_* X'$ (see Proposition 3) and there is a functorial morphism $Y \longrightarrow h_* X'$. The latter is an open immersion by Proposition 2.

In order to show that $Y \longrightarrow h_* X'$ is an equivalence of functors, it is enough to show that each functorial morphism $a : T \longrightarrow h_* X'$, where T is an arbitrary S-scheme, factors uniquely through Y or, what amounts to the same, that the latter is the case locally in a neighborhood of each point $z \in T$. Let (z_j) be the finite family of points in $T \times_S S'$ lying over z. Furthermore, let $a' : T \times_S S' \longrightarrow X'$ be the morphism corresponding to a, and set $x_j = a'(z_j)$. By our assumption, there is an affine open subscheme $U' \subset X'$ containing all points x_j. We know already that $h_* U'$ is representable by an S-scheme U and that the canonical morphism $U \longrightarrow h_* X'$ is an open immersion; the latter factors through Y by the definition of Y. Replacing T by a suitable open subscheme containing z, we may assume that $a' : T' \longrightarrow X'$ factors through U'. Then $a : T \longrightarrow h_* X'$ factors through U and, hence, through Y. The factorization is unique due to the fact that $Y \longrightarrow h_* X'$ is an open immersion. □

We want to mention some general properties of Weil restrictions, assuming that we are in the situation of Theorem 4.

Proposition 5. *Let $S' \longrightarrow S$ be a morphism of schemes which is finite and locally free, and let X' be an S'-scheme. Assume that the Weil restriction $X = \mathfrak{R}_{S'/S}(X')$ exists as an S-scheme, and consider the following properties for relative schemes:*

(a) *quasi-compact.*

(b) *separated,*

(c) *locally of finite type,*
(d) *locally of finite presentation,*
(e) *finite presentation,*
(f) *proper,*
(g) *flat,*
(h) *smooth.*

Then the above properties carry over from X' to X under the following additional assumptions:

property (a) *if S is locally noetherian or if $S' \longrightarrow S$ is étale,*
properties (b), (c), (d), (e), *and* (h) *without any further assumptions, and*
properties (f) *and* (g) *if $S' \longrightarrow S$ is étale.*

Proof. Let us begin with properties which carry over from X' to X without any additional assumptions, say with property (b). Since the Weil restriction of the diagonal morphism $X' \longrightarrow X' \times_{S'} X'$ yields the diagonal morphism $X \longrightarrow X \times_S X$ and since the Weil restriction respects closed immersions by Proposition 2, we see that X is separated if X' is separated.

Next, let us look at properties (c) and (d). That they carry over from X' to X follows from the construction of Weil restrictions in the affine case. Namely, if X' is a closed subscheme of the affine n-space $\mathbb{A}^n_{S'}$, and if $S' \longrightarrow S$ is a finite and free morphism of affine schemes, say of degree d, then it follows from Proposition 2 that X is a closed subscheme of $\mathfrak{R}_{S'/S}(\mathbb{A}^n_{S'}) \simeq \mathbb{A}^m_S$ where $m = nd$. So X is locally of finite type if the same is true for X'. Furthermore, the proof of Proposition 2 shows that the ideal defining X as a closed subscheme of $\mathbb{A}^m_S$ is finitely generated if the same is true for X' as a closed subscheme of $\mathbb{A}^n_{S'}$. So it follows that X is locally of finite presentation if the same is true for X'. The latter result can also be obtained by functorial arguments using the characterization [EGA IV_3], 8.14.2, of morphisms which are locally of finite presentation.

If X' satisfies property (e), we can view it as an S-scheme of finite presentation. Using a limit argument, we may assume that S is noetherian. Then X is locally of finite presentation, since property (d) carries over from X' to X, and quasi-compact over S since, as we will see below, also property (a) carries over from X' to X if the base S is noetherian. But then X is of finite presentation over S.

Finally, the characterization of smoothness in terms of the lifting property 2.2/6 shows by functorial reasons that X satisfies property (h) if X' does.

Now assume that $S' \longrightarrow S$ is étale and finite. In order to show that X satisfies properties (a), (f), or (g) if X' does, we may work locally on S, say in a neighborhood of a point $s \in S$. Furthermore, Weil restrictions commute with base change on S. So we may replace S by an étale neighborhood of s. But then, since locally up to étale base change étale morphisms are open immersions, see 2.3/8, we are reduced to the case where S' consists of a finite disjoint sum $\coprod S_i$ of copies S_i of S and where $S' \longrightarrow S$ is the canonical map. Then, in terms of fibred products over S,

$$\mathfrak{R}_{S'/S}(X') \simeq \prod_i \mathfrak{R}_{S_i/S}(X' \times_{S'} S_i) \simeq \prod_i X' \times_{S'} S_i \,,$$

and it is trivial that X satisfies properties (a), (f), or (g) if X' does.

It remains to show that, under appropriate conditions, property (a) carries over from X' to X, a fact which is already known if $S' \longrightarrow S$ is étale. We claim that it is also true for radicial morphisms. To verify this, it is enough to prove that, for S' radicial over S, the Weil restriction $\mathfrak{R}_{S'/S}$ transforms any affine open covering (U_j') of X' into an affine open covering $(\mathfrak{R}_{S'/S}(U_j'))$ of X. Looking at fibres over S, we may assume that S is the spectrum of a field K. Then S' consists of a finite-dimensional local K-algebra K' whose residue field is purely inseparable over K. Now let (U_j') be an affine open covering of X'. To see that the sets $\mathfrak{R}_{K'/K}(U_j')$ really cover X, consider a geometric point $\operatorname{Spec} E \longrightarrow X$ where E is a field over K. Then the scheme $\operatorname{Spec}(E \otimes_K K')$ consists of a single point and the corresponding morphism $\operatorname{Spec}(E \otimes_K K') \longrightarrow X'$ must factor through a member of the open covering (U_j') of X'. Consequently, $\operatorname{Spec} E \longrightarrow X$ factors through a member of the family $(\mathfrak{R}_{K'/K}(U_j'))$ which justifies our claim.

Now assume that the base S is locally noetherian. In order to show that X satisfies property (a) if X' does, we may assume that S is noetherian. We will conclude by using a noetherian argument and a stratification of S. Let η be a generic point of S. Restricting ourselves to a neighborhood of η, we can assume that S is irreducible and, since quasi-compactness can be tested after killing nilpotent elements of structure sheaves, that S is reduced. Furthermore, we can assume that S and S' are affine, say $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} R'$. The fibre S'_η is the spectrum of the finite-dimensional K-algebra $K' = R' \otimes_R K$ where $K = k(\eta) = Q(R)$. Let L be the maximal étale K-subalgebra between K and K'. It is obtained as follows. Decompose K' into a finite direct product $\prod K_i'$ of local K-algebras K_i' and, for each i, choose a maximal separable extension field L_i between K and K_i'. Then the residue field of K_i' is purely inseparable over L_i and we have $L = \prod L_i$. Set $T := \operatorname{Spec}(R' \cap L)$ so that $S' \longrightarrow S$ factors through T. Over the generic point η, the finite morphism $T \longrightarrow S$ is étale. Thus, using the openness of the étale locus, we know that $T \longrightarrow S$ is étale over an open neighborhood of η. Restricting to this neighborhood, we may assume that $T \longrightarrow S$ is étale everywhere. Furthermore, for each $a \in K'$, there is an integer n such that a^n belongs to L. This property carries over to the fibres of $S' \longrightarrow T$ so that the latter morphism is radicial. Since $X = \mathfrak{R}_{T/S}(\mathfrak{R}_{S'/T}(X'))$, we see by what we have proved above for étale and for radicial morphisms that, working over a neighborhood of η, the scheme X is quasi-compact if X' is.

The argument just given shows that the original morphism $X \longrightarrow S$ is quasi-compact over a dense open subset of S if X' is quasi-compact over S'. Looking at the complement S_1 of this set and viewing it as a scheme with respect to the canonical reduced structure, we can perform the base change $S_1 \longrightarrow S$. It follows in the same way that $X \times_S S_1 \longrightarrow S_1$ is quasi-compact over a dense open subset of S_1. Continuing this way, the procedure will stop after finitely many steps due to the noetherian hypothesis. Thus, finally, it is seen that X is quasi-compact over S. □

We want to add, again in the situation of Theorem 4, that, for any S-scheme X, the canonical morphism $X \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S')$ is a closed immersion, provided X and, thus, $\mathfrak{R}_{S'/S}(X \times_S S')$ are separated. This follows by means of descent from the fact that the composition of canonical morphisms

$$X \times_S S' \longrightarrow \mathfrak{R}_{S'/S}(X \times_S S') \times_S S' \longrightarrow X \times_S S'$$

is the identity on $X \times_S S'$.

Finally, let us state how Néron models behave with respect to Weil restrictions.

Proposition 6. *Let $S' \longrightarrow S$ be a finite and flat morphism of Dedekind schemes. Let* $\operatorname{Spec} K$ *and* $\operatorname{Spec} K'$ *denote the schemes of generic points of S and S'. Furthermore, consider a torsor X' (under a smooth S'-group scheme G') which is a Néron model of the scheme of generic fibres $X' \times_{S'} \operatorname{Spec} K'$. Then the Weil restriction $X = \mathfrak{R}_{S'/S}(X')$ exists as an S-scheme and is a Néron model of the scheme of generic fibres $X \times_S \operatorname{Spec} K$.*

Proof. Using the quasi-projectivity of torsors over Dedekind schemes (cf. 6.4/1), the existence of $X = \mathfrak{R}_{S'/S}(X')$ as an S-scheme follows from Theorem 4. Furthermore, it follows from Proposition 5 that X is separated, of finite type, and smooth. Finally, that X satisfies the Néron mapping property is a formal consequence of the definition of Weil restrictions, namely of the equation

$$\operatorname{Hom}_S(Z, X) = \operatorname{Hom}_{S'}(Z \times_S S', X') .$$

□